Contents 21

z-Transforms

Learning outcomes

In this Workbook you will learn about the properties and applications of the z-transform, a major mathematical tool for the analysis and design of discrete systems including digital control systems.

The z-Transform 21.1

Introduction

The z-transform is the major mathematical tool for analysis in such topics as digital control and digital signal processing. In this introductory Section we lay the foundations of the subject by briefly discussing sequences, shifting of sequences and difference equations. Readers familiar with these topics can proceed directly to Section 21.2 where z-transforms are first introduced.

Prerequisites

Before starting this Section you should ...

- have competence with algebra

Learning Outcomes

On completion you should be able to ...

- explain what is meant by a sequence and by a difference equation
- distinguish between first and second order difference equations
- shift sequences to the left or right

1. Preliminaries: Sequences and Difference Equations

Sequences

A sequence is a set of numbers formed according to some definite rule. For example the sequence

$$\{1, 4, 9, 16, 25, \ldots\} \qquad (1)$$

is formed by the **squares** of the positive integers.
If we write

$$y_1 = 1, \quad y_2 = 4, \quad y_3 = 9, \ldots$$

then the **general** or $\boldsymbol{n^{th}}$ **term** of the sequence (1) is $y_n = n^2$. The notations $y(n)$ and $y[n]$ are also used sometimes to denote the general term. The notation $\{y_n\}$ is used as an abbreviation for a whole sequence.

An alternative way of considering a sequence is to view it as being obtained by **sampling** a continuous function. In the above example the sequence of squares can be regarded as being obtained from the function

$$y(t) = t^2$$

by sampling the function at $t = 1, 2, 3, \ldots$ as shown in Figure 1.

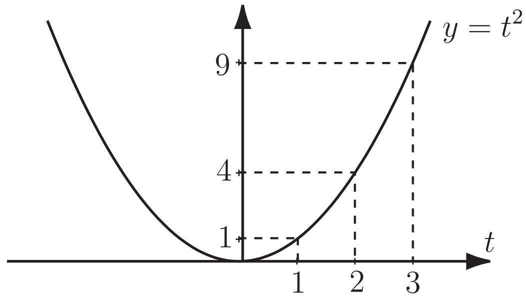


Figure 1

The notation $y(n)$, as opposed to y_n, for the general term of a sequence emphasizes this sampling aspect.

Task

Find the general term of the sequence $\{2, 4, 8, 16, 32, \ldots\}$.

Your solution

Answer

The terms of the sequence are the integer powers of 2: $y_1 = 2 = 2^1 \quad y_2 = 4 = 2^2$

$y_3 = 8 = 2^3 \ldots$ so $y_n = 2^n$.

Here the sequence $\{2^n\}$ are the sample values of the continuous function $y(t) = 2^t$ at $t = 1, 2, 3, \ldots$

An alternative way of defining a sequence is as follows:

(i) give the first term y_1 of the sequence
(ii) give the rule for obtaining the $(n+1)^{\text{th}}$ term from the n^{th}.

A simple example is

$$y_{n+1} = y_n + d \qquad y_1 = a$$

where a and d are constants.
It is straightforward to obtain an expression for y_n in terms of n as follows:

$$\begin{aligned}
y_2 &= y_1 + d = a + d \\
y_3 &= y_2 + d = a + d + d = a + 2d \\
y_4 &= y_3 + d = a + 3d \\
&\vdots \\
y_n &= a + (n-1)d
\end{aligned} \qquad (2)$$

This sequence characterised by a **constant difference** between successive terms

$$y_{n+1} - y_n = d \qquad n = 1, 2, 3, \ldots$$

is called an arithmetic sequence.

Task

Calculate the n^{th} term of the arithmetic sequence defined by

$$y_{n+1} - y_n = 2 \qquad y_1 = 9.$$

Write out the first 4 terms of this sequence explicitly.

Suggest why an arithmetic sequence is also known as a linear sequence.

Your solution

Answer

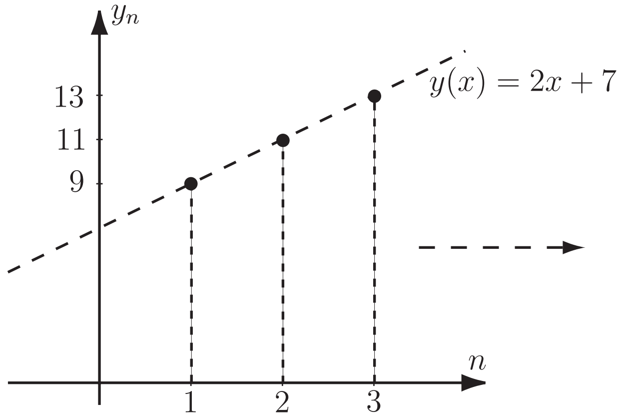
We have, using (2),

$y_n = 9 + (n-1)2$ or

$y_n = 2n + 7$

so $y_1 = 9$ (as given), $y_2 = 11,\ y_3 = 13,\ y_4 = 15, \ldots$

A graph of y_n against n would be just a set of points but all lie on the straight line $y = 2x + 7$, hence the term 'linear sequence'.



Nomenclature

The equation

$$y_{n+1} - y_n = d \tag{3}$$

is called a **difference equation** or **recurrence equation** or more specifically a first order, constant coefficient, linear, difference equation.
The sequence whose n^{th} term is

$$y_n = a + (n-1)d \tag{4}$$

is the **solution** of (3) for the **initial condition** $y_1 = a$.
The coefficients in (3) are the numbers preceding the terms y_{n+1} and y_n so are 1 and -1 respectively.
The classification **first** order for the difference equation (3) follows because the difference between the highest and lowest subscripts is $n + 1 - n = 1$.

Now consider again the sequence

$$\{y_n\} = \{2^n\}$$

Clearly

$$y_{n+1} - y_n = 2^{n+1} - 2^n = 2^n$$

so the difference here is dependent on n i.e. is not constant. Hence the sequence $\{2^n\} = \{2, 4, 8, \ldots\}$ is not an arithmetic sequence.

Task

For the sequence $\{y_n\} = 2^n$ calculate $y_{n+1} - 2y_n$. Hence write down a difference equation and initial condition for which $\{2^n\}$ is the solution.

Your solution

Answer

$$y_{n+1} - 2y_n = 2^{n+1} - 2 \times 2^n = 2^{n+1} - 2^{n+1} = 0$$

Hence $y_n = 2^n$ is the solution of the **homogeneous** difference equation

$$y_{n+1} - 2y_n = 0 \qquad (5)$$

with initial condition $y_1 = 2$.

The term 'homogeneous' refers to the fact that the right-hand side of the difference equation (5) is zero.

More generally it follows that

$$y_{n+1} - Ay_n = 0 \qquad y_1 = A$$

has solution sequence $\{y_n\}$ with general term

$$y_n = A^n$$

A second order difference equation

Second order difference equations are characterised, as you would expect, by a difference of 2 between the highest and lowest subscripts. A famous example of a constant coefficient second order difference equation is

$$y_{n+2} = y_{n+1} + y_n \quad \text{or} \quad y_{n+2} - y_{n+1} - y_n = 0 \qquad (6)$$

The solution $\{y_n\}$ of (6) is a sequence where any term is the sum of the two preceding ones.

Task

What additional information is needed if (6) is to be solved?

Your solution

Answer
Two initial conditions, the values of y_1 and y_2 must be specified so we can calculate

$$\begin{aligned} y_3 &= y_2 + y_1 \\ y_4 &= y_3 + y_2 \end{aligned}$$

and so on.

Task

Find the first 6 terms of the solution sequence of (6) for each of the following sets of initial conditions

(a) $y_1 = 1 \qquad y_2 = 3$

(b) $y_1 = 1 \qquad y_2 = 1$

Your solution

Answer
(a) $\{1, 3, 4, 7, 11, 18 \ldots\}$

(b) $\{1, 1, 2, 3, 5, 8, \ldots\}$ (7)

The sequence (7) is a very famous one; it is known as the **Fibonacci Sequence**. It follows that the solution sequence of the difference equation (6)

$$y_{n+1} = y_{n+1} + y_n$$

with initial conditions $y_1 = y_2 = 1$ is the Fibonacci sequence. What is not so obvious is what is the general term y_n of this sequence.

One way of obtaining y_n in this case, and for many other linear constant coefficient difference equations, is via a technique involving $Z-$transforms which we shall introduce shortly.

Shifting of sequences

Right Shift

Recall the sequence $\{y_n\} = \{n^2\}$ or, writing out the first few terms explicitly,

$$\{y_n\} = \{1, 4, 9, 16, 25, \ldots\}$$

The sequence $\{v_n\} = \{0, 1, 4, 9, 16, 25, \ldots\}$ contains the same numbers as y_n but they are all shifted one place to the right. The general term of this shifted sequence is

$$v_n = (n-1)^2 \qquad n = 1, 2, 3, \ldots$$

Similarly the sequence

$$\{w_n\} = \{0, 0, 1, 4, 9, 16, 25, \ldots\}$$

has general term

$$w_n = \begin{cases} (n-2)^2 & n = 2, 3, \ldots \\ 0 & n = 1 \end{cases}$$

Task

For the sequence $\{y_n\} = \{2^n\} = \{2, 4, 8, 16, \ldots\}$ write out explicitly the first 6 terms and the general terms of the sequences v_n and w_n obtained respectively by shifting the terms of $\{y_n\}$

(a) one place to the right (b) three places the the right.

Your solution

Answer

(a)

$$\{v_n\} = \{0, 2, 4, 8, 16, 32 \ldots\} \qquad v_n = \begin{cases} 2^{n-1} & n = 2, 3, 4, \ldots \\ 0 & n = 1 \end{cases}$$

(b)

$$\{w_n\} = \{0, 0, 0, 2, 4, 8 \ldots\} \qquad w_n = \begin{cases} 2^{n-3} & n = 4, 5, 6, \ldots \\ 0 & n = 1, 2, 3 \end{cases}$$

The operation of shifting the terms of a sequence is an important one in digital signal processing and digital control. We shall have more to say about this later. For the moment we just note that in a digital system a right shift can be produced by **delay unit** denoted symbolically as follows:

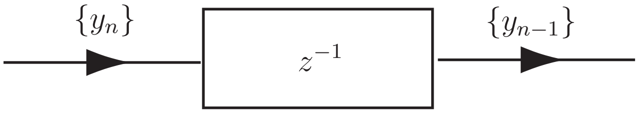


Figure 2

A shift of 2 units to the right could be produced by 2 such delay units in series:

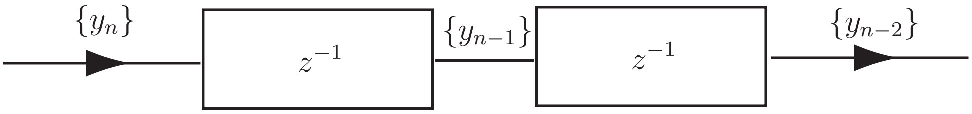


Figure 3

(The significance of writing z^{-1} will emerge later when we have studied $z-$transforms.)

Left Shift

Suppose we again consider the sequence of squares

$$\{y_n\} = \{1, 4, 9, 16, 25, \ldots\}$$

with $y_n = n^2$.

Shifting all the numbers one place to the **left** (or **advancing** the sequence) means that the sequence $\{v_n\}$ generated has terms

$$v_0 = y_1 = 1 \qquad v_1 = y_2 = 4 \qquad v_2 = y_3 = 9 \ldots$$

and so has general term

$$\begin{aligned} v_n &= (n+1)^2 \qquad n = 0, 1, 2, \ldots \\ &= y_{n+1} \end{aligned}$$

Notice here the appearance of the zero subscript for the first time.

Shifting the terms of $\{v_n\}$ one place to the left or equivalently the terms of $\{y_n\}$ two places to the left generates a sequence $\{w_n\}$ where

$$w_{-1} = v_0 = y_1 = 1 \qquad w_0 = v_1 = y_2 = 4$$

and so on.

The general term is

$$\begin{aligned} w_n &= (n+2)^2 \qquad n = -1, 0, 1, 2, \ldots \\ &= y_{n+2} \end{aligned}$$

Task

If $\{y_n\} = \{1, 1, 2, 3, 5, \ldots\}$ $\;n = 1, 2, 3, \ldots$ is the Fibonacci sequence, write out the terms of the sequences $\{y_{n+1}\},\ \{y_{n+2}\}$.

Your solution

Answer

$y_{n+1} = \{1, 1, 2, 3, 5, \ldots\}$ $\quad$ where $y_0 = 1$ (arrowed), $y_1 = 1,\ y_2 = 2, \ldots$
$\qquad\qquad\qquad\uparrow$

$y_{n+2} = \{1, 1, 2, 3, 5, \ldots\}$ $\quad$ where $y_{-1} = 1,\ \ y_0 = 1$ (arrowed), $y_1 = 2,\ y_2 = 3, \ldots$
$\qquad\qquad\qquad\quad\uparrow$

It should be clear from this discussion of left shifted sequences that the simpler idea of a sequence 'beginning' at $n = 1$ and containing only terms $y_1, y_2, \ldots$ has to be modified.

We should instead think of a sequence as **two-sided** i.e. $\{y_n\}$ defined for **all** integer values of n and zero. In writing out the 'middle' terms of a two sided sequence it is convenient to show by an arrow the term y_0.

For example the sequence $\{y_n\} = \{n^2\}$ $\quad n = 0, \pm 1, \pm 2, \ldots$ could be written

$\{\ldots 9, 4, 1, 0, 1, 4, 9, \ldots\}$
$\qquad\qquad\quad\uparrow$

A sequence which is zero for negative integers n is sometimes called a **causal** sequence.
For example the sequence, denoted by $\{u_n\}$,

$$u_n = \begin{cases} 0 & n = -1, -2, -3, \ldots \\ 1 & n = 0, 1, 2, 3, \ldots \end{cases}$$

is causal. Figure 4 makes it clear why $\{u_n\}$ is called the **unit step sequence**.

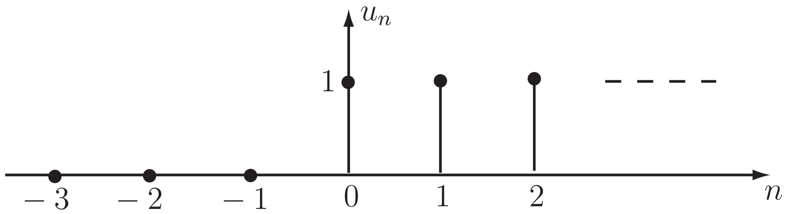


Figure 4

The 'curly bracket' notation for the unit step sequence with the $n = 0$ term arrowed is

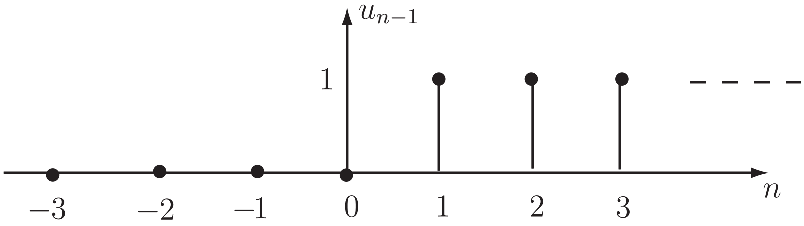
$\{u_n\} = \{\ldots, 0, 0, 0, 1, 1, 1, \ldots\}$
$\qquad\qquad\qquad\qquad\ \uparrow$

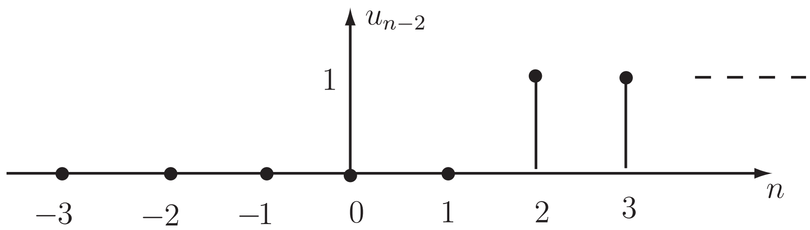
Task

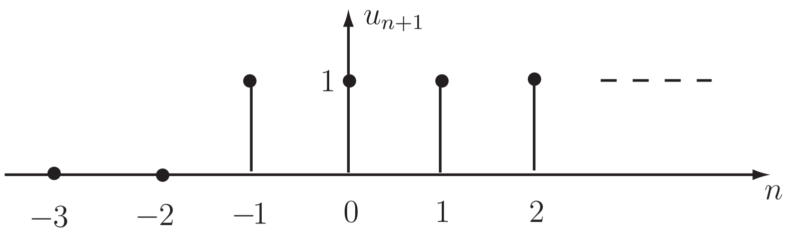
Draw graphs of the sequences $\{u_{n-1}\}$, $\{u_{n-2}\}$, $\{u_{n+1}\}$ where $\{u_n\}$ is the unit step sequence.

Your solution

Answer







Basics of z-Transform Theory

21.2

Introduction

In this Section, which is absolutely fundamental, we define what is meant by the z-transform of a sequence. We then obtain the z-transform of some important sequences and discuss useful properties of the transform.

Most of the results obtained are tabulated at the end of the Section.

The z-transform is the major mathematical tool for analysis in such areas as digital control and digital signal processing.

Prerequisites

Before starting this Section you should ...

- understand sigma (Σ) notation for summations
- be familiar with geometric series and the binomial theorem
- have studied basic complex number theory including complex exponentials

Learning Outcomes

On completion you should be able to ...

- define the z-transform of a sequence
- obtain the z-transform of simple sequences from the definition or from basic properties of the z-transform

1. The z-transform

If you have studied the Laplace transform either in a Mathematics course for Engineers and Scientists or have applied it in, for example, an analog control course you may recall that

1. the Laplace transform definition involves an **integral**
2. applying the Laplace transform to certain ordinary differential equations turns them into simpler (algebraic) equations
3. use of the Laplace transform gives rise to the basic concept of the **transfer function** of a continuous (or analog) system.

The z-transform plays a similar role for **discrete** systems, i.e. ones where **sequences** are involved, to that played by the Laplace transform for systems where the basic variable t is continuous. Specifically:

1. the z-transform definition involves a **summation**
2. the z-transform converts certain difference equations to algebraic equations
3. use of the z-transform gives rise to the concept of the transfer function of discrete (or digital) systems.

Key Point 1

Definition:

For a sequence $\{y_n\}$ the z-transform denoted by $Y(z)$ is given by the **infinite series**

$$Y(z) = y_0 + y_1 z^{-1} + y_2 z^{-2} + \ldots = \sum_{n=0}^{\infty} y_n z^{-n} \qquad (1)$$

Notes:

1. The z-transform only involves the terms y_n, $n = 0, 1, 2, \ldots$ of the sequence. Terms $y_{-1}, y_{-2}, \ldots$ whether zero or non-zero, are not involved.
2. The infinite series in (1) **must converge** for $Y(z)$ to be defined as a precise function of z. We shall discuss this point further with specific examples shortly.
3. The precise significance of the quantity (strictly the 'variable') z need not concern us except to note that it is complex and, unlike n, is continuous.

Key Point 2

We use the notation $\mathbb{Z}\{y_n\} = Y(z)$ to mean that the z-transform of the sequence $\{y_n\}$ is $Y(z)$.

Less strictly one might write $\mathbb{Z}y_n = Y(z)$. Some texts use the notation $y_n \leftrightarrow Y(z)$ to denote that (the sequence) y_n and (the function) $Y(z)$ form a z-transform pair.

We shall also call $\{y_n\}$ the **inverse z-transform** of $Y(z)$ and write symbolically

$$\{y_n\} = \mathbb{Z}^{-1}Y(z).$$

2. Commonly used z-transforms

Unit impulse sequence (delta sequence)

This is a simple but important sequence denoted by δ_n and defined as

$$\delta_n = \begin{cases} 1 & n = 0 \\ 0 & n = \pm 1, \pm 2, \ldots \end{cases}$$

The significance of the term 'unit impulse' is obvious from this definition.
By the definition (1) of the z-transform

$$\begin{aligned} \mathbb{Z}\{\delta_n\} &= 1 + 0z^{-1} + 0z^{-2} + \ldots \\ &= 1 \end{aligned}$$

If the single non-zero value is other than at $n = 0$ the calculation of the z-transform is equally simple. For example,

$$\delta_{n-3} = \begin{cases} 1 & n = 3 \\ 0 & \text{otherwise} \end{cases}$$

From (1) we obtain

$$\begin{aligned} \mathbb{Z}\{\delta_{n-3}\} &= 0 + 0z^{-1} + 0z^{-2} + z^{-3} + 0z^{-4} + \ldots \\ &= z^{-3} \end{aligned}$$

Task

Write down the definition of δ_{n-m} where m is any positive integer and obtain its z-transform.

Your solution

Answer

$$\delta_{n-m} = \begin{cases} 1 & n = m \\ 0 & \text{otherwise} \end{cases} \qquad \mathbb{Z}\{\delta_{n-m}\} = z^{-m}$$

Key Point 3

$$\mathbb{Z}\{\delta_{n-m}\} = z^{-m} \qquad m = 0, 1, 2, \ldots$$

Unit step sequence

As we saw earlier in this Workbook the unit step sequence is

$$u_n = \begin{cases} 1 & n = 0, 1, 2, \ldots \\ 0 & n = -1, -2, -3, \ldots \end{cases}$$

Then, by the definition (1)

$$\mathbb{Z}\{u_n\} = 1 + 1z^{-1} + 1z^{-2} + \ldots$$

The infinite series here is a geometric series (with a constant ratio z^{-1} between successive terms). Hence the sum of the first N terms is

$$\begin{aligned} S_N &= 1 + z^{-1} + \ldots + z^{-(N-1)} \\ &= \frac{1 - z^{-N}}{1 - z^{-1}} \end{aligned}$$

As $N \to \infty \quad S_N \to \dfrac{1}{1 - z^{-1}}$ provided $|z^{-1}| < 1$

Hence, in what is called the **closed form** of this z-transform we have the result given in the following Key Point:

Key Point 4

$$\mathbb{Z}\{u_n\} = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1} \equiv U(z) \text{ say}, \qquad |z^{-1}| < 1$$

The restriction that this result is only valid if $|z^{-1}| < 1$ or, equivalently $|z| > 1$ means that the position of the complex quantity z must lie outside the circle centre origin and of unit radius in an Argand diagram. This restriction is not too significant in elementary applications of the z-transform.

The geometric sequence $\{a^n\}$

Task

For any arbitrary constant a obtain the z-transform of the causal sequence

$$f_n = \begin{cases} 0 & n = -1, -2, -3, \ldots \\ a^n & n = 0, 1, 2, 3, \ldots \end{cases}$$

Your solution

Answer

We have, by the definition in Key Point 1,

$$F(z) = \mathbb{Z}\{f_n\} = 1 + az^{-1} + a^2z^{-2} + \ldots$$

which is a geometric series with common ratio az^{-1}. Hence, provided $|az^{-1}| < 1$, the closed form of the z-transform is

$$F(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}.$$

The z-transform of this sequence $\{a^n\}$, which is itself a geometric sequence is summarized in Key Point 5.

Key Point 5

$$\mathbb{Z}\{a^n\} = \frac{1}{1 - az^{-1}} = \frac{z}{z - a} \qquad |z| > |a|.$$

Notice that if $a = 1$ we recover the result for the z-transform of the unit step sequence.

Task

Use Key Point 5 to write down the z-transform of the following causal sequences

(a) 2^n

(b) $(-1)^n$, the unit alternating sequence

(c) e^{-n}

(d) $e^{-\alpha n}$ where α is a constant.

Your solution

Answer

(a) Using $a = 2$ $\qquad \mathbb{Z}\{2^n\} = \dfrac{1}{1 - 2z^{-1}} = \dfrac{z}{z-2} \qquad |z| > 2$

(b) Using $a = -1$ $\qquad \mathbb{Z}\{(-1)^n\} = \dfrac{1}{1 + z^{-1}} = \dfrac{z}{z+1} \qquad |z| > 1$

(c) Using $a = e^{-1}$ $\qquad \mathbb{Z}\{e^{-n}\} = \dfrac{z}{z - e^{-1}} \qquad |z| > e^{-1}$

(d) Using $a = e^{-\alpha}$ $\qquad \mathbb{Z}\{e^{-\alpha n}\} = \dfrac{z}{z - e^{-\alpha}} \qquad |z| > \mathrm{e}^{-\alpha}$

The basic z-transforms obtained have all been straightforwardly found from the definition in Key Point 1. To obtain further useful results we need a knowledge of some of the properties of z-transforms.

3. Linearity property and applications

Linearity property

This simple property states that if $\{v_n\}$ and $\{w_n\}$ have z-transforms $V(z)$ and $W(z)$ respectively then

$$\mathbb{Z}\{av_n + bw_n\} = aV(z) + bW(z)$$

for any constants a and b.
(In particular if $a = b = 1$ this property tells us that adding sequences corresponds to adding their z-transforms).

The proof of the linearity property is straightforward using obvious properties of the summation operation. By the z-transform definition:

$$\begin{aligned}
\mathbb{Z}\{av_n + bw_n\} &= \sum_{n=0}^{\infty}(av_n + bw_n)z^{-n} \\
&= \sum_{n=0}^{\infty}(av_n z^{-n} + bw_n z^{-n}) \\
&= a\sum_{n=0}^{\infty} v_n z^{-n} + b\sum_{n=0}^{\infty} w_n z^{-n} \\
&= aV(z) + bV(z)
\end{aligned}$$

We can now use the linearity property and the exponential sequence $\{e^{-\alpha n}\}$ to obtain the z-transforms of hyperbolic and of trigonometric sequences relatively easily. For example,

$$\sinh n = \frac{e^n - e^{-n}}{2}$$

Hence, by the linearity property,

$$\begin{aligned}
\mathbb{Z}\{\sinh n\} &= \frac{1}{2}\mathbb{Z}\{e^n\} - \frac{1}{2}\mathbb{Z}\{e^{-n}\} \\
&= \frac{1}{2}\left(\frac{z}{z-e} - \frac{z}{z-e^{-1}}\right) \\
&= \frac{z}{2}\left(\frac{z - e^{-1} - (z-e)}{z^2 - (e+e^{-1})z + 1}\right) \\
&= \frac{z}{2}\left(\frac{e - e^{-1}}{z^2 - (2\cosh 1)z + 1}\right) \\
&= \frac{z\sinh 1}{z^2 - 2z\cosh 1 + 1}
\end{aligned}$$

Using αn instead of n in this calculation, where α is a constant, we obtain

$$\mathbb{Z}\{\sinh \alpha n\} = \frac{z\sinh\alpha}{z^2 - 2z\cosh\alpha + 1}$$

Task

Using $\cosh \alpha n \equiv \dfrac{e^{\alpha n} + e^{-\alpha n}}{2}$ obtain the z-transform of the sequence $\{\cosh \alpha n\} = \{1,\ \cosh\alpha,\ \cosh 2\alpha, \ldots\}$

Your solution

Answer

We have, by linearity,

$$\begin{aligned}
\mathbb{Z}\{\cosh \alpha n\} &= \frac{1}{2}\mathbb{Z}\{e^{\alpha n}\} + \frac{1}{2}\mathbb{Z}\{e^{-\alpha n}\} \\
&= \frac{z}{2}\left(\frac{1}{z - e^{\alpha}} + \frac{1}{z - e^{-\alpha}}\right) \\
&= \frac{z}{2}\left(\frac{2z - (e^{\alpha} + e^{-\alpha})}{z^2 - 2z\cosh\alpha + 1}\right) \\
&= \frac{z^2 - z\cosh\alpha}{z^2 - 2z\cosh\alpha + 1}
\end{aligned}$$

Trigonometric sequences

If we use the result

$$\mathbb{Z}\{a^n\} = \frac{z}{z-a} \qquad |z| > |a|$$

with, respectively, $a = e^{\mathrm{i}\omega}$ and $a = e^{-\mathrm{i}\omega}$ where ω is a constant and i denotes $\sqrt{-1}$ we obtain

$$\mathbb{Z}\{e^{\mathrm{i}\omega n}\} = \frac{z}{z - e^{+\mathrm{i}\omega}} \qquad \mathbb{Z}\{e^{-\mathrm{i}\omega n}\} = \frac{z}{z - e^{-\mathrm{i}\omega}}$$

Hence, recalling from complex number theory that

$$\cos x = \frac{e^{\mathrm{i}x} + e^{-\mathrm{i}x}}{2}$$

we can state, using the linearity property, that

$$\begin{aligned}
\mathbb{Z}\{\cos \omega n\} &= \frac{1}{2}\mathbb{Z}\{e^{\mathrm{i}\omega n}\} + \frac{1}{2}\mathbb{Z}\{e^{-\mathrm{i}\omega n}\} \\
&= \frac{z}{2}\left(\frac{1}{z - e^{\mathrm{i}\omega}} + \frac{1}{z - e^{-\mathrm{i}\omega}}\right) \\
&= \frac{z}{2}\left(\frac{2z - (e^{\mathrm{i}\omega} + e^{-\mathrm{i}\omega})}{z^2 - (e^{\mathrm{i}\omega} + e^{-\mathrm{i}\omega})z + 1}\right) \\
&= \frac{z^2 - z\cos\omega}{z^2 - 2z\cos\omega + 1}
\end{aligned}$$

(Note the similarity of the algebra here to that arising in the corresponding hyperbolic case. Note also the similarity of the results for $\mathbb{Z}\{\cosh \alpha n\}$ and $\mathbb{Z}\{\cos \omega n\}$.)

Task

By a similar procedure to that used above for $\mathbb{Z}\{\cos \omega n\}$ obtain $\mathbb{Z}\{\sin \omega n\}$.

Your solution

Answer

We have

$$\mathbb{Z}\{\sin \omega n\} = \frac{1}{2\mathrm{i}}\mathbb{Z}\{e^{\mathrm{i}\omega n}\} - \frac{1}{2\mathrm{i}}\mathbb{Z}\{e^{-\mathrm{i}\omega n}\} \qquad \text{(Don't miss the i factor here!)}$$

$$\begin{aligned} \therefore \qquad \mathbb{Z}\{\sin \omega n\} &= \frac{z}{2\mathrm{i}}\left(\frac{1}{z-e^{\mathrm{i}\omega}} - \frac{1}{z-e^{-\mathrm{i}\omega}}\right) \\ &= \frac{z}{2\mathrm{i}}\left(\frac{-e^{-\mathrm{i}\omega}+e^{\mathrm{i}\omega}}{z^2-2z\cos\omega+1}\right) \\ &= \frac{z\sin\omega}{z^2-2z\cos\omega+1} \end{aligned}$$

Key Point 6

$$\begin{aligned} \mathbb{Z}\{\cos \omega n\} &= \frac{z^2 - z\cos\omega}{z^2-2z\cos\omega+1} \\ \mathbb{Z}\{\sin \omega n\} &= \frac{z\sin\omega}{z^2-2z\cos\omega+1} \end{aligned}$$

Notice the same denominator in the two results in Key Point 6.

Key Point 7

$$\begin{aligned} \mathbb{Z}\{\cosh \alpha n\} &= \frac{z^2 - z\cosh\alpha}{z^2-2z\cosh\alpha+1} \\ \mathbb{Z}\{\sinh \alpha n\} &= \frac{z\sinh\alpha}{z^2-2z\cosh\alpha+1} \end{aligned}$$

Again notice the denominators in Key Point 7. Compare these results with those for the two trigonometric sequences in Key Point 6.

Task

Use Key Points 6 and 7 to write down the z-transforms of

(a) $\left\{\sin \dfrac{n}{2}\right\}$ (b) $\{\cos 3n\}$ (c) $\{\sinh 2n\}$ (d) $\{\cosh n\}$

Your solution

Answer

(a) $$\mathbb{Z}\left\{\sin \frac{n}{2}\right\} = \frac{z \sin\left(\frac{1}{2}\right)}{z^2 - 2z\cos\left(\frac{1}{2}\right) + 1}$$

(b) $$\mathbb{Z}\{\cos 3n\} = \frac{z^2 - z\cos 3}{z^2 - 2z\cos 3 + 1}$$

(c) $$\mathbb{Z}\{\sinh 2n\} = \frac{z \sinh 2}{z^2 - 2z\cosh 2 + 1}$$

(d) $$\mathbb{Z}\{\cosh n\} = \frac{z^2 - z\cosh 1}{z^2 - 2z\cosh 1 + 1}$$

Task

Use the results for $\mathbb{Z}\{\cos \omega n\}$ and $\mathbb{Z}\{\sin \omega n\}$ in Key Point 6 to obtain the z-transforms of

(a) $\{\cos(n\pi)\}$ (b) $\left\{\sin\left(\frac{n\pi}{2}\right)\right\}$ (c) $\left\{\cos\left(\frac{n\pi}{2}\right)\right\}$

Write out the first few terms of each sequence.

Your solution

Answer

(a) With $\omega = \pi$

$$\mathbb{Z}\{\cos n\pi\} = \frac{z^2 - z\cos\pi}{z^2 - 2z\cos\pi + 1} = \frac{z^2 + z}{z^2 + 2z + 1} = \frac{z}{z+1}$$

$$\{\cos n\pi\} = \{1, -1, 1, -1, \ldots\} = \{(-1)^n\}$$

We have re-derived the z-transform of the unit alternating sequence. (See Task on page 17).

(b) With $\omega = \frac{\pi}{2}$

$$\mathbb{Z}\left\{\sin\frac{n\pi}{2}\right\} = \frac{z\sin\left(\frac{\pi}{2}\right)}{z^2 - 2z\cos\left(\frac{\pi}{2}\right) + 1} = \frac{z}{z^2+1}$$

where $\left\{\sin\frac{n\pi}{2}\right\} = \{0, 1, 0, -1, 0, \ldots\}$

(c) With $\omega = \frac{\pi}{2}$ $\quad \mathbb{Z}\left\{\cos\frac{n\pi}{2}\right\} = \frac{z^2 - \cos\left(\frac{\pi}{2}\right)}{z^2+1} = \frac{z^2}{z^2+1}$

where $\left\{\cos\frac{n\pi}{2}\right\} = \{1, 0, -1, 0, 1, \ldots\}$

(These three results can also be readily obtained from the definition of the z-transform. Try!)

4. Further z-transform properties

We showed earlier that the results

$\mathbb{Z}\{v_n + w_n\} = V(z) + W(z)$ and similarly $\mathbb{Z}\{v_n - w_n\} = V(z) - W(z)$

follow from the linearity property.

You should be clear that there is **no** comparable result for the **product** of two sequences.

$\mathbb{Z}\{v_n w_n\}$ **is not equal to** $V(z)W(z)$

For two specific products of sequences however we can derive useful results.

Multiplication of a sequence by a^n

Suppose f_n is an arbitrary sequence with z-transform $F(z)$.
Consider the sequence $\{v_n\}$ where

$v_n = a^n f_n$ i.e. $\{v_0, v_1, v_2, \ldots\} = \{f_0, af_1, a^2 f_2, \ldots\}$

By the z-transform definition

$$\begin{aligned}
\mathbb{Z}\{v_n\} &= v_0 + v_1 z^{-1} + v_2 z^{-2} + \ldots \\
&= f_0 + a\ f_1 z^{-1} + a^2 f_2 z^{-2} + \ldots \\
&= \sum_{n=0}^{\infty} a^n f_n z^{-n} \\
&= \sum_{n=0}^{\infty} f_n \left(\frac{z}{a}\right)^{-n} \\
\text{But } F(z) &= \sum_{n=0}^{\infty} f_n z^{-n}
\end{aligned}$$

Thus we have shown that $\mathbb{Z}\{a^n f_n\} = F\left(\dfrac{z}{a}\right)$

Key Point 8

$$\mathbb{Z}\{a^n f_n\} = F\left(\frac{z}{a}\right)$$

That is, multiplying a sequence $\{f_n\}$ by the sequence $\{a^n\}$ does not change the form of the z-transform $F(z)$. We merely replace z by $\dfrac{z}{a}$ in that transform.

For example, using Key Point 6 we have

$$\mathbb{Z}\{\cos n\} = \frac{z^2 - z\cos 1}{z^2 - 2z\cos 1 + 1}$$

So, replacing z by $\dfrac{z}{(\frac{1}{2})} = 2z$,

$$\mathbb{Z}\left\{\left(\frac{1}{2}\right)^n \cos n\right\} = \frac{(2z)^2 - (2z)\cos 1}{(2z)^2 - 4z\cos 1 + 1}$$

Task

Using Key Point 8, write down the z-transform of the sequence $\{v_n\}$ where

$$v_n = e^{-2n}\sin 3n$$

Your solution

Answer

We have, $\quad \mathbb{Z}\{\sin 3n\} = \dfrac{z\sin 3}{z^2 - 2z\cos 3 + 1}$

so with $a = e^{-2}$ we replace z by $z\ e^{+2}$ to obtain

$$\begin{aligned}\mathbb{Z}\{v_n\} = \mathbb{Z}\{e^{-2n}\sin 3n\} &= \frac{ze^2\sin 3}{(ze^2)^2 - 2ze^2\cos 3 + 1}\\ &= \frac{ze^{-2}\sin 3}{z^2 - 2ze^{-2}\cos 3 + e^{-4}}\end{aligned}$$

Using the property just discussed write down the z-transform of the sequence $\{w_n\}$ where

$$w_n = e^{-\alpha n}\cos\omega n$$

Your solution

Answer

We have, $\mathbb{Z}\{\cos\omega n\} = \dfrac{z^2 - z\cos\omega}{z^2 - 2z\cos\omega + 1}$

So replacing z by ze^{α} we obtain

$$\begin{aligned}\mathbb{Z}\{w_n\} = \mathbb{Z}\{e^{-\alpha n}\cos\omega n\} &= \frac{(ze^{\alpha})^2 - ze^{\alpha}\cos\omega}{(ze^{\alpha})^2 - 2ze^{\alpha}\cos\omega + 1}\\ &= \frac{z^2 - ze^{-\alpha}\cos\omega}{z^2 - 2ze^{-\alpha}\cos\omega + e^{-2\alpha}}\end{aligned}$$

Key Point 9

$$\begin{aligned}\mathbb{Z}\{e^{-\alpha n}\cos\omega n\} &= \frac{z^2 - ze^{-\alpha}\cos\omega}{z^2 - 2ze^{-\alpha}\cos\omega + e^{-2\alpha}}\\ \mathbb{Z}\{e^{-\alpha n}\sin\omega n\} &= \frac{ze^{-\alpha}\sin\omega}{z^2 - 2ze^{-\alpha}\cos\omega + e^{-2\alpha}}\end{aligned}$$

Note the same denominator in each case.

Multiplication of a sequence by n

An important sequence whose z-transform we have not yet obtained is the **unit ramp** sequence $\{r_n\}$:

$$r_n = \begin{cases} 0 & n = -1, -2, -3, \ldots \\ n & n = 0, 1, 2, \ldots \end{cases}$$

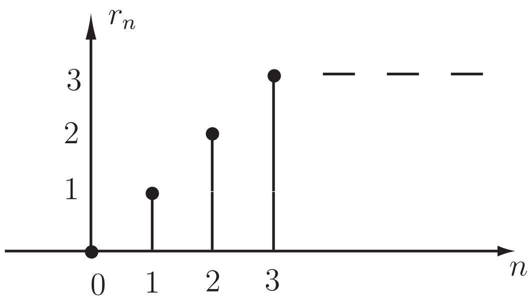


Figure 5

Figure 5 clearly suggests the nomenclature 'ramp'.

We shall attempt to use the z-transform of $\{r_n\}$ from the definition:

$$\mathbb{Z}\{r_n\} = 0 + 1z^{-1} + 2z^{-2} + 3z^{-3} + \ldots$$

This is not a geometric series but we can write

$$\begin{aligned} z^{-1} + 2z^{-2} + 3z^{-3} &= z^{-1}(1 + 2z^{-1} + 3z^{-2} + \ldots) \\ &= z^{-1}(1 - z^{-1})^{-2} \qquad |z^{-1}| < 1 \end{aligned}$$

where we have used the binomial theorem (HELM 16.3) .
Hence

$$\begin{aligned} \mathbb{Z}\{r_n\} = \mathbb{Z}\{n\} &= \frac{1}{z\left(1 - \frac{1}{z}\right)^2} \\ &= \frac{z}{(z-1)^2} \qquad |z| > 1 \end{aligned}$$

Key Point 10

The z-transform of the unit ramp sequence is

$$\mathbb{Z}\{r_n\} = \frac{z}{(z-1)^2} = R(z) \quad \text{(say)}$$

Recall now that the unit step sequence has z-transform $\mathbb{Z}\{u_n\} = \dfrac{z}{(z-1)} = U(z)$ (say) which is the subject of the next Task.

Task

Obtain the derivative of $U(z) = \dfrac{z}{(z-1)}$ with respect to z.

Your solution

Answer

We have, using the quotient rule of differentiation:

$$\frac{dU}{dz} = \frac{d}{dz}\left(\frac{z}{z-1}\right) = \frac{(z-1)1-(z)(1)}{(z-1)^2}$$

$$= \frac{-1}{(z-1)^2}$$

We also know that

$$R(z) = \frac{z}{(z-1)^2} = (-z)\left(-\frac{1}{(z-1)^2}\right) = -z\frac{dU}{dz} \qquad (3)$$

Also, if we compare the sequences

$$\begin{array}{l} u_n = \{0, 0, 1, 1, 1, 1, \ldots\} \\ \qquad\qquad\quad \uparrow \\ r_n = \{0, 0, 0, 1, 2, 3, \ldots\} \\ \qquad\qquad\quad \uparrow \end{array}$$

we see that $r_n = n\ u_n$, (4)

so from (3) and (4) we conclude that $\mathbb{Z}\{n\ u_n\} = -z\dfrac{dU}{dz}$

Now let us consider the problem more generally.

Let f_n be an arbitrary sequence with z-transform $F(z)$:

$$F(z) = f_0 + f_1 z^{-1} + f_2 z^{-2} + f_3 z^{-3} + \ldots = \sum_{n=0}^{\infty} f_n z^{-n}$$

We differentiate both sides with respect to the variable z, doing this term-by-term on the right-hand side. Thus

$$\begin{aligned}\frac{dF}{dz} &= -f_1z^{-2} - 2f_2z^{-3} - 3f_3z^{-4} - \ldots = \sum_{n=1}^{\infty}(-n)f_nz^{-n-1} \\ &= -z^{-1}(f_1z^{-1} + 2f_2z^{-2} + 3f_3z^{-3} + \ldots) = -z^{-1}\sum_{n=1}^{\infty} n\ f_nz^{-n}\end{aligned}$$

But the bracketed term is the z-transform of the sequence

$$\{n\ f_n\} = \{0, f_1, 2f_2, 3f_3, \ldots\}$$

Thus if $F(z) = \mathbb{Z}\{f_n\}$ we have shown that

$$\frac{dF}{dz} = -z^{-1}\mathbb{Z}\{n\ f_n\} \quad \text{or} \quad \mathbb{Z}\{n\ f_n\} = -z\frac{dF}{dz}$$

We have already (equations (3) and (4) above) demonstrated this result for the case $f_n = u_n$.

Key Point 11

If $\quad \mathbb{Z}\{f_n\} = F(z) \quad$ then $\quad \mathbb{Z}\{n\ f_n\} = -z\dfrac{dF}{dz}$

Task

By differentiating the z-transform $R(z)$ of the unit ramp sequence obtain the z-transform of the causal sequence $\{n^2\}$.

Your solution

Answer
We have

$$\mathbb{Z}\{n\} = \frac{z}{(z-1)^2}$$

so

$$\mathbb{Z}\{n^2\} = \mathbb{Z}\{n.n\} = -z\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right)$$

By the quotient rule

$$\begin{aligned}\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right) &= \frac{(z-1)^2-(z)(2)(z-1)}{(z-1)^4}\\ &= \frac{z-1-2z}{(z-1)^3} = \frac{-1-z}{(z-1)^3}\end{aligned}$$

Multiplying by $-z$ we obtain

$$\mathbb{Z}\{n^2\} = \frac{z+z^2}{(z-1)^3} = \frac{z(1+z)}{(z-1)^3}$$

Clearly this process can be continued to obtain the transforms of $\{n^3\},\ \{n^4\},\dots$ etc.

5. Shifting properties of the z-transform

In this subsection we consider perhaps the most important properties of the z-transform. These properties relate the z-transform $Y(z)$ of a sequence $\{y_n\}$ to the z-transforms of

(i) **right shifted** or **delayed sequences** $\{y_{n-1}\}\{y_{n-2}\}$ etc.
(ii) **left shifted** or **advanced sequences** $\{y_{n+1}\},\{y_{n+2}\}$ etc.

The results obtained, formally called shift theorems, are vital in enabling us to solve certain types of difference equation and are also invaluable in the analysis of digital systems of various types.

Right shift theorems

Let $\{v_n\} = \{y_{n-1}\}$ i.e. the terms of the sequence $\{v_n\}$ are the same as those of $\{y_n\}$ but shifted one place to the right. The z-transforms are, by definition,

$$\begin{aligned}Y(z) &= y_0 + y_1z^{-1} + y_2z^{-2} + y_jz^{-3} + \dots\\ V(z) &= v_0 + v_1z^{-1} + v_2z^{-2} + v_3z^{-3} + \dots\\ &= y_{-1} + y_0z^{-1} + y_1z^{-2} + y_2z^{-3} + \dots\\ &= y_{-1} + z^{-1}(y_0 + y_1z^{-1} + y_2z^{-2} + \dots)\end{aligned}$$

i.e.

$$V(z) = \mathbb{Z}\{y_{n-1}\} = y_{-1} + z^{-1}Y(z)$$

Task

Obtain the z-transform of the sequence $\{w_n\} = \{y_{n-2}\}$ using the method illustrated above.

Your solution

Answer

The z-transform of $\{w_n\}$ is $\quad W(z) = w_0 + w_1 z^{-1} + w_2 z^{-2} + w_3 z^{-3} + \ldots \quad$ or, since $w_n = y_{n-2}$,

$$\begin{aligned} W(z) &= y_{-2} + y_{-1}z^{-1} + y_0 z^{-2} + y_1 z^{-3} + \ldots \\ &= y_{-2} + y_{-1}z^{-1} + z^{-2}(y_0 + y_1 z^{-1} + \ldots) \end{aligned}$$

i.e. $W(z) = \mathbb{Z}\{y_{n-2}\} = y_{-2} + y_{-1}z^{-1} + z^{-2}Y(z)$

Clearly, we could proceed in a similar way to obtains a general result for $\mathbb{Z}\{y_{n-m}\}$ where m is any positive integer. The result is

$$\mathbb{Z}\{y_{n-m}\} = y_{-m} + y_{-m+1}z^{-1} + \ldots + y_{-1}z^{-m+1} + z^{-m}Y(z)$$

For the particular case of causal sequences (where $y_{-1} = y_{-2} = \ldots = 0$) these results are particularly simple:

$$\left.\begin{aligned} \mathbb{Z}\{y_{n-1}\} &= z^{-1}Y(z) \\ \mathbb{Z}\{y_{n-2}\} &= z^{-2}Y(z) \\ &\ \vdots \\ \mathbb{Z}\{y_{n-m}\} &= z^{-m}Y(z) \end{aligned}\right\} \text{(causal systems only)}$$

You may recall from earlier in this Workbook that in a digital system we represented the right shift operation symbolically in the following way:

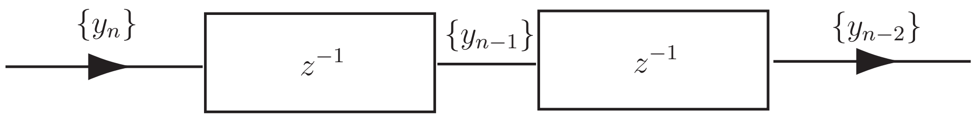


Figure 6

The significance of the z^{-1} factor inside the rectangles should now be clearer. If we replace the 'input' and 'output' sequences by their z-transforms:

$$\mathbb{Z}\{y_n\} = Y(z) \qquad \mathbb{Z}\{y_{n-1}\} = z^{-1}Y(z)$$

it is evident that in the z-transform 'domain' the shift becomes a multiplication by the factor z^{-1}.
N.B. This discussion applies strictly only to causal sequences.

Notational point:
A causal sequence is sometimes written as $y_n u_n$ where u_n is the unit step sequence

$$u_n = \begin{cases} 0 & n = -1, -2, \ldots \\ 1 & n = 0, 1, 2, \ldots \end{cases}$$

The right shift theorem is then written, for a causal sequence,

$$\mathbb{Z}\{y_{n-m}u_{n-m}\} = z^{-m}Y(z)$$

Examples

Recall that the z-transform of the causal sequence $\{a^n\}$ is $\dfrac{z}{z-a}$. It follows, from the right shift theorems that

(i) $\mathbb{Z}\{a^{n-1}\} = \mathbb{Z}\{0, 1, a, a^2, \ldots\} = \dfrac{zz^{-1}}{z-a} = \dfrac{1}{z-a}$

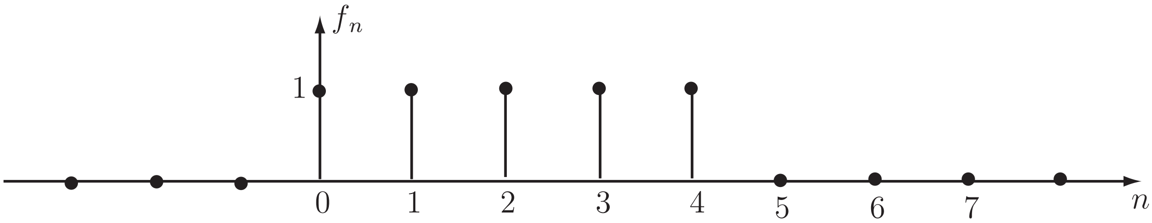
$\uparrow$

(ii) $\mathbb{Z}\{a^{n-2}\} = \mathbb{Z}\{0, 0, 1, a, a^2, \ldots\} = \dfrac{z^{-1}}{z-a} = \dfrac{1}{z(z-a)}$

$\uparrow$

Task

Write the following sequence f_n as a difference of two unit step sequences. Hence obtain its z-transform.



Your solution

Answer

Since $\{u_n\} = \begin{cases} 1 & n = 0, 1, 2, \ldots \\ 0 & n = -1, -2, \ldots \end{cases}$

and $\{u_{n-5}\} = \begin{cases} 1 & n = 5, 6, 7, \ldots \\ 0 & \text{otherwise} \end{cases}$

it follows that

$$f_n = u_n - u_{n-5}$$

Hence $\quad F(z) = \dfrac{z}{z-1} - \dfrac{z^{-5}z}{z-1} = \dfrac{z - z^{-4}}{z-1}$

Left shift theorems

Recall that the sequences $\{y_{n+1}\}$, $\{y_{n+2}\} \ldots$ denote the sequences obtained by shifting the sequence $\{y_n\}$ by $1, 2, \ldots$ units to the left respectively. Thus, since $Y(z) = \mathbb{Z}\{y_n\} = y_0 + y_1 z^{-1} + y_2 z^{-2} + \ldots$ then

$$\begin{aligned} \mathbb{Z}\{y_{n+1}\} &= y_1 + y_2 z^{-1} + y_3 z^{-2} + \ldots \\ &= y_1 + z(y_2 z^{-2} + y_3 z^{-3} + \ldots) \end{aligned}$$

The term in brackets is the z-transform of the unshifted sequence $\{y_n\}$ apart from its first two terms: thus

$$\mathbb{Z}\{y_{n+1}\} = y_1 + z(Y(z) - y_0 - y_1 z^{-1})$$

$$\therefore \qquad Z\{y_{n+1}\} = zY(z) - zy_0$$

Task

Obtain the z-transform of the sequence $\{y_{n+2}\}$ using the method illustrated above.

Your solution

Answer

$$
\begin{aligned}
\mathbb{Z}\{y_{n+2}\} &= y_2 + y_3 z^{-1} + y_4 z^{-2} + \ldots \\
&= y_2 + z^2(y_3 z^{-3} + y_4 z^{-4} + \ldots) \\
&= y_2 + z^2(Y(z) - y_0 - y_1 z^{-1} - y_2 z^{-2})
\end{aligned}
$$

$$\therefore \qquad \mathbb{Z}\{y_{n+2}\} = z^2 Y(z) - z^2 y_0 - z y_1$$

These left shift theorems have simple forms in special cases:

if $y_0 = 0 \qquad \mathbb{Z}\{y_{n+1}\} = z\ Y(z)$

if $y_0 = y_1 = 0 \qquad \mathbb{Z}\{y_{n+2}\} = z^2 Y(z)$

if $y_0 = y_1 = \ldots y_{m-1} = 0 \qquad \mathbb{Z}\{y_{n+m}\} = z^m Y(z)$

Key Point 12

The **right shift theorems** or delay theorems are:

$$
\begin{aligned}
\mathbb{Z}\{y_{n-1}\} &= y_{-1} + z^{-1} Y(z) \\
\mathbb{Z}\{y_{n-2}\} &= y_{-2} + y_{-1} z^{-1} + z^{-2} Y(z) \\
\vdots \quad & \quad \vdots \quad \vdots \qquad \vdots \\
\mathbb{Z}\{y_{n-m}\} &= y_{-m} + y_{-m+1} z^{-1} + \ldots + y_{-1} z^{-m+1} + z^{-m} Y(z)
\end{aligned}
$$

The **left shift theorems** or advance theorems are:

$$
\begin{aligned}
\mathbb{Z}\{y_{n+1}\} &= zY(z) - z y_0 \\
\mathbb{Z}\{y_{n+2}\} &= z^2 Y(z) - z^2 y_0 - z y_1 \\
\vdots \quad & \quad \vdots \\
\mathbb{Z}\{y_{n-m}\} &= z^m Y(z) - z^m y_0 - z^{m-1} y_1 - \ldots - z y_{m-1}
\end{aligned}
$$

Note carefully the occurrence of **positive** powers of z in the **left** shift theorems and of **negative** powers of z in the **right** shift theorems.

Table 1: z-transforms

f_n	$F(z)$	Name
δ_n	1	unit impulse
δ_{n-m}	z^{-m}	
u_n	$\dfrac{z}{z-1}$	unit step sequence
a^n	$\dfrac{z}{z-a}$	geometric sequence
$e^{\alpha n}$	$\dfrac{z}{z-e^{\alpha}}$	
$\sinh \alpha n$	$\dfrac{z \sinh \alpha}{z^2 - 2z\cosh \alpha + 1}$	
$\cosh \alpha n$	$\dfrac{z^2 - z\cosh \alpha}{z^2 - 2z\cosh \alpha + 1}$	
$\sin \omega n$	$\dfrac{z \sin \omega}{z^2 - 2z\cos \omega + 1}$	
$\cos \omega n$	$\dfrac{z^2 - z\cos \omega}{z^2 - 2z\cos \omega + 1}$	
$e^{-\alpha n}\sin \omega n$	$\dfrac{ze^{-\alpha}\sin \omega}{z^2 - 2ze^{-\alpha}\cos \omega + e^{-2\alpha}}$	
$e^{-\alpha n}\cos \omega n$	$\dfrac{z^2 - ze^{-\alpha}\cos \omega}{z^2 - 2ze^{-\alpha}\cos \omega + e^{-2\alpha}}$	
n	$\dfrac{z}{(z-1)^2}$	ramp sequence
n^2	$\dfrac{z(z+1)}{(z-1)^3}$	
n^3	$\dfrac{z(z^2+4z+1)}{(z-1)^4}$	
$a^n f_n$	$F\left(\dfrac{z}{a}\right)$	
$n\ f_n$	$-z\dfrac{dF}{dz}$	

This table has been copied to the back of this Workbook (page 96) for convenience.

z-Transforms and Difference Equations 21.3

Introduction

In this we apply z-transforms to the solution of certain types of difference equation. We shall see that this is done by turning the difference equation into an ordinary algebraic equation. We investigate both first and second order difference equations.

A key aspect in this process in the inversion of the z-transform. As well as demonstrating the use of partial fractions for this purpose we show an alternative, often easier, method using what are known as residues.

Prerequisites

Before starting this Section you should ...

- have studied carefully Section 21.2
- be familiar with simple partial fractions

Learning Outcomes

On completion you should be able to ...

- invert z-transforms using partial fractions or residues where appropriate
- solve constant coefficient linear difference equations using z-transforms

1. Solution of difference equations using z-transforms

Using z-transforms, in particular the shift theorems discussed at the end of the previous Section, provides a useful method of solving certain types of difference equation. In particular linear constant coefficient difference equations are amenable to the z-transform technique although certain other types can also be tackled. In fact all the difference equations that we looked at in Section 21.1 were linear:

$$y_{n+1} = y_n + d \qquad \text{(1st order)}$$
$$y_{n+1} = A\, y_n \qquad \text{(1st order)}$$
$$y_{n+2} = y_{n+1} + y_n \qquad \text{(2nd order)}$$

Other examples of linear difference equations are

$$y_{n+2} + 4y_{n+1} - 3y_n = n^2 \qquad \text{(2nd order)}$$
$$y_{n+1} + y_n = n\, 3^n \qquad \text{(1st order)}$$

The key point is that for a difference equation to be classified as linear the terms of the sequence $\{y_n\}$ arise only to power 1 or, more precisely, the highest subscript term is obtainable as a **linear combination** of the lower ones. All the examples cited above are consequently linear. Note carefully that the term n^2 in our fourth example does not imply non-linearity since linearity is determined by the y_n terms.

Examples of non-linear difference equations are

$$\begin{aligned} y_{n+1} &= \sqrt{y_n + 1} \\ y_{n+1}^2 + 2\, y_n &= 3 \\ y_{n+1} y_n &= n \\ \cos(y_{n+1}) &= y_n \end{aligned}$$

We shall not consider the problem of solving non-linear difference equations.

The five linear equations listed above also have **constant coefficients**; for example:

$$y_{n+2} + 4y_{n+1} - 3y_n = n^2$$

has the constant coefficients $1,\ 4,\ -3$.

The (linear) difference equation

$$n\, y_{n+2} - y_{n+1} + y_n = 0$$

has one variable coefficient viz n and so is not classified as a constant coefficient difference equation.

Solution of first order linear constant coefficient difference equations

Consider the first order difference equation

$$y_{n+1} - 3y_n = 4 \qquad n = 0, 1, 2, \ldots$$

The equation could be solved in a step-by-step or **recursive** manner, provided that y_0 is known because

$$y_1 = 4 + 3y_0 \qquad y_2 = 4 + 3y_1 \qquad y_3 = 4 + 3y_2 \quad \text{and so on.}$$

This process will certainly produce the terms of the solution sequence $\{y_n\}$ but the general term y_n may not be obvious.

So consider

$$y_{n+1} - 3y_n = 4 \qquad n = 0, 1, 2, \ldots \tag{1}$$

with initial condition $y_0 = 1$.

We multiply both sides of (1) by z^{-n} and sum each side over all positive integer values of n and zero. We obtain

$$\sum_{n=0}^{\infty} (y_{n+1} - 3y_n) z^{-n} = \sum_{n=0}^{\infty} 4z^{-n}$$

or

$$\sum_{n=0}^{\infty} y_{n+1} z^{-n} - 3 \sum_{n=0}^{\infty} y_n z^{-n} = 4 \sum_{n=0}^{\infty} z^{-n} \tag{2}$$

The three terms in (2) are clearly recognisable as z-transforms.

The right-hand side is the z-transform of the constant sequence $\{4,\ 4, \ldots\}$ which is $\dfrac{4z}{z-1}$.

If $Y(z) = \displaystyle\sum_{n=0}^{\infty} y_n z^{-n}$ denotes the z-transform of the sequence $\{y_n\}$ that we are seeking then $\displaystyle\sum_{n=0}^{\infty} y_{n+1} z^{-n} = z\, Y(z) - zy_0$ (by the left shift theorem).

Consequently (2) can be written

$$z\, Y(z) - zy_0 - 3\, Y(z) = \frac{4z}{z-1} \tag{3}$$

Equation (3) is the z-transform of the original difference equation (1). The intervening steps have been included here for explanation purposes but we shall omit them in future. The important point is that (3) is no longer a difference equation. It is an **algebraic** equation where the unknown, $Y(z)$, is the z-transform of the solution sequence $\{y_n\}$.

We now insert the initial condition $y_0 = 1$ and solve (3) for $Y(z)$:

$$\begin{aligned} (z-3)Y(z) - z &= \frac{4z}{(z-1)} \\ (z-3)Y(z) &= \frac{4z}{z-1} + z = \frac{z^2+3z}{z-1} \end{aligned}$$

so $$Y(z) = \frac{z^2+3z}{(z-1)(z-3)} \tag{4}$$

The final step consists of obtaining the sequence $\{y_n\}$ of which (4) is the z-transform. As it stands (4) is not recognizable as any of the standard transforms that we have obtained. Consequently, one method of 'inverting' (4) is to use a **partial fraction** expansion. (We assume that you are familiar with simple partial fractions. See HELM 3.6)

Thus

$$\begin{aligned} Y(z) &= z\frac{(z+3)}{(z-1)(z-3)} \\ &= z\left(\frac{-2}{z-1}+\frac{3}{z-3}\right) \qquad \text{(in partial fractions)} \end{aligned}$$

so $\quad Y(z) = \dfrac{-2z}{z-1} + \dfrac{3z}{z-3}$

Now, taking inverse z-transforms, the general term y_n is, using the linearity property,

$$y_n = -2\mathbb{Z}^{-1}\{\frac{z}{z-1}\} + 3\ \mathbb{Z}^{-1}\{\frac{z}{z-3}\}$$

The symbolic notation $\mathbb{Z}^{-1}$ is common and is short for 'the inverse z-transform of'.

Task

Using standard z-transforms write down y_n explicitly, where

$$y_n = -2\mathbb{Z}^{-1}\{\frac{z}{z-1}\} + 3\ \mathbb{Z}^{-1}\{\frac{z}{z-3}\}$$

Your solution

Answer

$$y_n = -2 + 3 \times 3^n = -2 + 3^{n+1} \qquad n = 0, 1, 2, \ldots \qquad (5)$$

Checking the solution:

From this solution (5)

$$y_n = -2 + 3^{n+1}$$

we easily obtain

$$y_0 = -2 + 3 = 1 \qquad \text{(as given)}$$

$$y_1 = -2 + 3^2 = 7$$

$$y_2 = -2 + 3^3 = 25$$

$$y_3 = -2 + 3^4 = 79 \quad \text{etc.}$$

These agree with those obtained by recursive solution of the given problem (1):

$$y_{n+1} - 3y_n = 4 \qquad y_0 = 1$$

which yields

$$y_1 = 4 + 3y_0 = 7$$

$$y_2 = 4 + 3y_1 = 25$$

$$y_3 = 4 + 3y_2 = 79 \qquad \text{etc.}$$

More conclusively we can put the solution (5) back into the left-hand side of the difference equation (1).

If $y_n = -2 + 3^{n+1}$

then $\quad 3y_n = -6 + 3^{n+2}$

and $\quad y_{n+1} = -2 + 3^{n+2}$

So, on the left-hand side of (1),

$$y_{n+1} - 3y_n = -2 + 3^{n+2} - (-6 + 3^{n+2})$$

which does indeed equal 4, the given right-hand side, and so the solution has been verified.

Key Point 13

To solve a linear constant coefficient difference equation, three steps are involved:

1. Replace each term in the difference equation by its z-transform and insert the initial condition(s).
2. Solve the resulting algebraic equation. (Thus gives the z-transform $Y(z)$ of the solution sequence.)
3. Find the inverse z-transform of $Y(z)$.

The third step is usually the most difficult. We will consider the problem of finding inverse z-transforms more fully later.

Task

Solve the difference equation

$$y_{n+1} - y_n = d \qquad n = 0, 1, 2, \ldots \qquad y_0 = a \tag{6}$$

where a and d are constants.
(The solution will give the n th term of an arithmetic sequence with a constant difference d and initial term a.)

Start by replacing each term of (6) by its z-transform:

Your solution

Answer
If $Y(z) = \mathbb{Z}\{y_n\}$ we obtain the algebraic equation

$$z\,Y(z) - zy_0 - Y(z) = \frac{d \times z}{(z-1)}$$

Note that the right-hand side transform is that of a constant sequence $\{d, d, \ldots\}$. Note also the use of the left shift theorem.

Now insert the initial condition $y_0 = a$ and then solve for $Y(z)$:

Your solution

Answer

$$(z-1)Y(z) = \frac{d \times z}{(z-1)} + z \times a$$
$$Y(z) = \frac{d \times z}{(z-1)^2} + \frac{a \times z}{z-1}$$

Finally take the inverse z-transform of the right-hand side. [Hint: Recall the z-transform of the ramp sequence $\{n\}$.]

Your solution

Answer
We have

$$y_n = d \times \mathbb{Z}^{-1}\{\frac{z}{(z-1)^2}\} + a \times \mathbb{Z}^{-1}\{\frac{z}{z-1}\}$$

$$\therefore \qquad y_n = dn + a \qquad n = 0, 1, 2, \ldots \qquad (7)$$

using the known z-transforms of the ramp and unit step sequences. Equation (7) may well be a familiar result to you – an arithmetic sequence whose 'zeroth' term is $y_0 = a$ has general term $y_n = a + nd$.

i.e. $\{y_n\} = \{a,\ a+d, \ldots a+nd, \ldots\}$

This solution is of course readily obtained by direct recursive solution of (6) without need for z-transforms. In this case the general term $(a + nd)$ is readily seen from the form of the recursive solution: (Make sure you really do see it).

N.B. If the term a is labelled as the **first** term (rather than the zeroth) then

$$y_1 = a,\ y_2 = a + d,\ y_3 - a + 2d,$$

so in this case the n th term is

$$y_n = a + (n-1)d$$

rather than (7).

Use of the right shift theorem in solving difference equations

The problem just solved was given by (6), i.e.

$$y_{n+1} - y_n = d \qquad \text{with } y_0 = a \qquad n = 0, 1, 2, \ldots$$

We obtained the solution

$$y_n = a + nd \qquad n = 0, 1, 2, \ldots$$

Now consider the problem

$$y_n - y_{n-1} = d \qquad n = 0, 1, 2, \ldots \tag{8}$$

with $y_{-1} = a$.
The only difference between the two problems is that the 'initial condition' in (8) is given at $n = -1$ rather than at $n = 0$. Writing out the first few terms should make this clear.

$$\begin{array}{cc}
(6) & (8) \\
y_1 - y_0 = d & y_0 - y_{-1} = d \\
y_2 - y_1 = d & y_1 - y_0 = d \\
\vdots & \vdots \\
y_{n+1} - y_n = d & y_n - y_{n-1} = d \\
y_0 = a & y_{-1} = a
\end{array}$$

The solution to (8) must therefore be the same as for (6) but with every term in the solution (7) of (6) shifted 1 unit to the left.
Thus the solution to (8) is expected to be

$$y_n = a + (n+1)d \qquad n = -1, 0, 1, 2, \ldots$$

(replacing n by $(n+1)$ in the solution (7)).

Task

Use the right shift theorem of z-transforms to solve (8) with the initial condition $y_{-1} = a$.

(a) Begin by taking the z-transform of (8), inserting the initial condition and solving for $Y(z)$:

Your solution

Answer
We have, for the z-transform of (8)

$$\begin{aligned} Y(z) - (z^{-1}Y(z) + y_{-1}) &= \frac{dz}{z-1} \qquad \text{[Note that here } dz \text{ means } d \times z] \\ Y(z)(1 - z^{-1}) - a &= \frac{dz}{z-1} \\ Y(z)\left(\frac{z-1}{z}\right) &= \frac{dz}{(z-1)} + a \\ Y(z) &= \frac{dz^2}{(z-1)^2} + \frac{az}{z-1} \end{aligned} \qquad (9)$$

The second term of $Y(z)$ has the inverse z-transform $\{a\ u_n\} = \{a, a, a, \ldots\}$.
The first term is less straightforward. However, we have already reasoned that the other term in y_n here should be $(n+1)d$.

(b) Show that the z-transform of $(n+1)d$ is $\dfrac{dz^2}{(z-1)^2}$. Use the standard transform of the ramp and step:

Your solution

Answer
We have

$$\mathbb{Z}\{(n+1)d\} = d\mathbb{Z}\{n\} + d\mathbb{Z}\{1\}$$

by the linearity property

$$\begin{aligned} \therefore \qquad \mathbb{Z}\{(n+1)d\} &= \frac{dz}{(z-1)^2} + \frac{dz}{z-1} \\ &= dz\left(\frac{1+z-1}{(z-1)^2}\right) \\ &= \frac{dz^2}{(z-1)^2} \end{aligned}$$

as expected.

(c) Finally, state y_n:

Your solution

Answer
Returning to (9) the inverse z-transform is

$$y_n = (n+1)d + a\ u_n \qquad \text{i.e.} \qquad y_n = a + (n+1)d \qquad n = -1, 0, 1, 2, \ldots$$

as we expected.

Task

Earlier in this Section (pages 37-39) we solved

$$y_{n+1} - 3y_n = 4 \qquad n = 0, 1, 2, \ldots \qquad \text{with } y_0 = 1.$$

Now solve $y_n - 3y_{n-1} = 4 \qquad n = 0, 1, 2, \ldots \qquad$ with $y_{-1} = 1$. (10)

Begin by obtaining the z-transform of y_n:

Your solution

Answer
We have, taking the z-transform of (10),

$$Y(z) - 3(z^{-1}Y(z) + 1) = \frac{4z}{z-1}$$

(using the right shift property and inserting the initial condition.)

$$\therefore \qquad Y(z) - 3z^{-1}Y(z) = 3 + \frac{4z}{z-1}$$

$$Y(z)\frac{(z-3)}{z} = 3 + \frac{4z}{z-1} \qquad \text{so} \qquad Y(z) = \frac{3z}{z-3} + \frac{4z^2}{(z-1)(z-3)}$$

Write the second term as $4z\left(\dfrac{z}{(z-1)(z-3)}\right)$ and obtain the partial fraction expansion of the bracketed term. Then complete the z-transform inversion.

Your solution

Answer

$$\frac{z}{(z-1)(z-3)} = \frac{-\frac{1}{2}}{z-1} + \frac{\frac{3}{2}}{z-3}$$

We now have

$$Y(z) = \frac{3z}{z-3} - \frac{2z}{z-1} + \frac{6z}{z-3}$$

so

$$y_n = 3 \times 3^n - 2 + 6 \times 3^n = -2 + 9 \times 3^n = -2 + 3^{n+2} \qquad (11)$$

Compare this solution (11) to that of the previous problem (5) on page 39:

Your solution

Answer

Solution (11) is just the solution sequence (5) moved 1 unit to the left. We anticipated this since the difference equation (10) and associated initial condition is the same as the difference equation (1) but shifted one unit to the left.

2. Second order difference equations

You will learn in this section about solving second order linear constant coefficient difference equations. In this case **two** initial conditions are required, typically either y_0 and y_1 or y_{-1} and y_{-2}. In the first case we use the left shift property of the z-transform, in the second case we use the right shift property. The same three basic steps are involved as in the first order case.

Task

By solving

$$y_{n+2} = y_{n+1} + y_n \qquad (12)$$

$$y_0 = y_1 = 1$$

obtain the general term y_n of the Fibonacci sequence.

Begin by taking the z-transform of (12), using the left shift property. Then insert the initial conditions and solve the resulting algebraic equation for $Y(z)$, the z-transform of $\{y_n\}$:

Your solution

Answer

$$z^2Y(z) - z^2y_0 - zy_1 = zY(z) - zy_0 + Y(z) \qquad \text{(taking z-transforms)}$$

$$z^2Y(z) - z^2 - z = zY(z) - z + Y(z) \qquad \text{(inserting initial conditions)}$$

$$(z^2 - z - 1)Y(z) = z^2$$

so

$$Y(z) = \frac{z^2}{z^2 - z - 1} \qquad \text{(solving for } Y(z)\text{).}$$

Now solve the quadratic equation $z^2 - z - 1 = 0$ and hence factorize the denominator of $Y(z)$:

Your solution

Answer

$z^2 - z - 1 = 0$

$$\therefore \qquad z = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

so if $a = \dfrac{1+\sqrt{5}}{2}, \quad b = \dfrac{1-\sqrt{5}}{2}$

$$Y(z) = \frac{z^2}{(z-a)(z-b)}$$

This form for $Y(z)$ often arises in solving second order difference equations. Write it in partial fractions and find y_n, leaving a and b as general at this stage:

Your solution

Answer

$$Y(z) = z\left(\frac{z}{(z-a)(z-b)}\right) = \frac{Az}{z-a} + \frac{Bz}{(z-b)} \qquad \text{in partial fractions}$$

where $A = \dfrac{a}{a-b}$ and $B = \dfrac{b}{b-a}$

Hence, taking inverse z-transforms

$$y_n = Aa^n + Bb^n = \frac{1}{(a-b)}\left(a^{n+1} - b^{n+1}\right) \qquad (13)$$

Now complete the Fibonacci problem:

Your solution

Answer

With $a = \dfrac{1+\sqrt{5}}{2}$ $\qquad b = \dfrac{1-\sqrt{5}}{2}$ $\qquad$ so $a - b = \sqrt{5}$

we obtain, using (13)

$$y_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n}\right) \qquad n = 2, 3, 4, \ldots$$

for the n th term of the Fibonacci sequence.

With an appropriate computational aid you could (i) check that this formula does indeed give the familiar sequence

$$\{1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ \ldots\}$$

and (ii) obtain, for example, y_{50} and y_{100}.

Key Point 14

The inverse z-transform of

$$Y(z) = \frac{z^2}{(z-a)(z-b)} \qquad a \neq b \qquad \text{is} \qquad y_n = \frac{1}{(a-b)}\left(a^{n+1} - b^{n+1}\right)$$

Task

Use the right shift property of z-transforms to solve the second order difference equation

$$y_n - 7y_{n-1} + 10\,y_{n-2} = 0 \quad \text{with } y_{-1} = 16 \quad \text{and } y_{-2} = 5.$$

[Hint: the steps involved are the same as in the previous Task]

Your solution

Answer

$$Y(z) - 7(z^{-1}Y(z) + 16) \quad + \quad 10(z^{-2}Y(z) + 16z^{-1} + 5) = 0$$

$$Y(z)(1 - 7z^{-1} + 10z^{-2}) \quad - \quad 112 + 160z^{-1} + 50 = 0$$

$$Y(z)\left(\frac{z^2 - 7z + 10}{z^2}\right) \quad = \quad 62 - 160z^{-1}$$

$$\begin{aligned} Y(z) &= \frac{62z^2}{z^2 - 7z + 10} - \frac{160z}{z^2 - 7z + 10} \\ &= z\frac{(62z - 160)}{(z-2)(z-5)} \\ &= \frac{12z}{z-2} + \frac{50z}{z-5} \qquad \text{in partial fractions} \\ \text{so} \qquad y_n &= 12 \times 2^n + 50 \times 5^n \qquad n = 0, 1, 2, \ldots \end{aligned}$$

We now give an Example where a quadratic equation with repeated solutions arises.

Example 1

(a) Obtain the z-transform of $\{f_n\} = \{na^n\}$.

(b) Solve

$$y_n - 6y_{n-1} + 9y_{n-2} = 0 \qquad n = 0, 1, 2, \ldots$$
$$y_{-1} = 1 \qquad y_{-2} = 0$$

[Hint: use the result from (a) at the inversion stage.]

Solution

(a) $Z\{n\} = \dfrac{z}{(z-1)^2} \qquad \therefore \quad Z\{na^n\} = \dfrac{z/a}{(z/a-1)^2} = \dfrac{az}{(z-a)^2}$ where we have used the property $Z\{f_n\, a^n\} = F\left(\dfrac{z}{a}\right)$

(b) Taking the z-transform of the difference equation and inserting the initial conditions:

$$Y(z) - 6(z^{-1}Y(z) + 1) + 9(z^{-2}Y(z) + z^{-1}) = 0$$

$$Y(z)(1 - 6z^{-1} + 9z^{-2}) = 6 - 9z^{-1}$$

$$Y(z)(z^2 - 6z + 9) = 6z^2 - 9z$$

$$Y(z) = \frac{6z^2 - 9z}{(z-3)^2} = z\left(\frac{6z-9}{(z-3)^2}\right) = z\left(\frac{6}{z-3} + \frac{9}{(z-3)^2}\right) \qquad \text{in partial fractions}$$

from which, using the result (a) on the second term,

$$y_n = 6 \times 3^n + 3n \times 3^n = (6 + 3n)3^n$$

We shall re-do this inversion by an alternative method shortly.

Task

Solve the difference equation

$$y_{n+2} + y_n = 0 \quad \text{with } y_0,\ y_1 \text{ arbitrary.} \qquad (14)$$

Start by obtaining $Y(z)$ using the left shift theorem:

Your solution

Answer

$$\begin{aligned} z^2Y(z) - z^2y_0 - zy_1 + Y(z) &= 0 \\ (z^2+1)Y(z) &= z^2y_0 + zy_1 \\ Y(z) &= \frac{z^2}{z^2+1}y_0 + \frac{z}{z^2+1}y_1 \end{aligned}$$

To find the inverse z-transforms recall the results for $Z\{\cos\omega n\}$ and $Z\{\sin\omega n\}$ from Key Point 6 (page 21) and some of the particular cases discussed in Section 21.2. Hence find y_n here:

Your solution

Answer

Taking $Z\{\cos\omega n\}$ and $Z\{\sin\omega n\}$ with $\omega = \dfrac{\pi}{2}$

$$\begin{aligned} Z\left\{\cos\left(\frac{n\pi}{2}\right)\right\} &= \frac{z^2}{z^2+1} \\ Z\left\{\sin\left(\frac{n\pi}{2}\right)\right\} &= \frac{z}{z^2+1} \end{aligned}$$

Hence $\quad y_n = y_0\mathbb{Z}^{-1}\{\dfrac{z^2}{z^2+1}\} + y_1\mathbb{Z}^{-1}\{\dfrac{z}{z^2+1}\} = y_0\cos\left(\dfrac{n\pi}{2}\right) + y_1\sin\left(\dfrac{n\pi}{2}\right) \qquad (15)$

Those of you who are familiar with **differential** equations may know that

$$\frac{d^2y}{dt^2} + y = 0 \qquad y(0) = y_0, \quad y'(0) = y_0' \qquad (16)$$

has solutions $y_1 = \cos t$ and $y_2 = \sin t$ and a general solution

$$y = c_1\cos t + c_2\sin t \qquad (17)$$

where $c_1 = y_0$ and $c_2 = y_0'$.

This differential equation is a model for simple harmonic oscillations. The difference equation (14) and its solution (15) are the discrete counterparts of (16) and (17).

3. Inversion of z-transforms using residues

This method has its basis in a branch of mathematics called complex integration. You may recall that the 'z' quantity of z-transforms is a complex quantity, more specifically a **complex** variable. However, it is not necessary to delve deeply into the theory of complex variables in order to obtain simple inverse z-transforms using what are called **residues**. In many cases inversion using residues is easier than using partial fractions. Hence reading on is strongly advised.

Pole of a function of a complex variable

If $G(z)$ is a function of the complex variable z and if

$$G(z) = \frac{G_1(z)}{(z - z_0)^k}$$

where $G_1(z_0)$ is non-zero and finite then $G(z)$ is said to have a **pole** of order k at $z = z_0$.
For example if

$$G(z) = \frac{6(z-2)}{z(z-3)(z-4)^2}$$

then $G(z)$ has the following 3 poles.

(i) pole of order 1 at $z = 0$
(ii) pole of order 1 at $z = 3$
(iii) pole of order 2 at $z = 4$.

(Poles of order 1 are sometimes known as simple poles.)

Note that when $z = 2$, $G(z) = 0$. Hence $z = 2$ is said to be a **zero** of $G(z)$. (It is the only zero in this case).

Task

Write down the poles and zeros of

$$G(z) = \frac{3(z+4)}{z^2(2z+1)(3z-9)} \qquad (18)$$

State the order of each pole.

Your solution

Answer

$G(z)$ has a zero when $z = -4$.
$G(z)$ has first order poles at $z = -1/2$, $z = 3$.
$G(z)$ has a second order pole at $z = 0$.

Residue at a pole

The residue of a complex function $G(z)$ at a first order pole z_0 is

$$\text{Res } (G(z),\ z_0) = [G(z)(z - z_0)]_{z_0} \tag{19}$$

The residue at a second order pole z_0 is

$$\text{Res } (G(z),\ z_0) = \left[\frac{d}{dz}(G(z)(z - z_0)^2)\right]_{z_0} \tag{20}$$

You need not worry about how these results are obtained or their full mathematical significance. (Any textbook on Complex Variable Theory could be consulted by interested readers.)

Example

Consider again the function (18) in the previous guided exercise.

$$\begin{aligned} G(z) &= \frac{3(z+4)}{z^2(2z+1)(3z-9)} \\ &= \frac{(z+4)}{2z^2\left(z+\frac{1}{2}\right)(z-3)} \end{aligned}$$

The second form is the more convenient for the residue formulae to be used.
Using (19) at the two first order poles:

$$\begin{aligned} \text{Res } \left(G(z),\ -\frac{1}{2}\right) &= \left[G(z)\left(z - \left(-\frac{1}{2}\right)\right)\right]_{\frac{1}{2}} \\ &= \left[\frac{(z+4)}{2z^2(z-3)}\right]_{\frac{1}{2}} = -\frac{18}{5} \\ \text{Res } [G(z),\ 3] &= \left[\frac{(z+4)}{2z^2\left(z+\dfrac{1}{2}\right)}\right]_3 = \frac{1}{9} \end{aligned}$$

Using (20) at the second order pole

$$\text{Res } (G(z),\ 0) = \left[\frac{d}{dz}(G(z)(z-0)^2)\right]_0$$

The differentiation has to be carried out before the substitution of $z = 0$ of course.

$$\begin{aligned} \therefore \qquad \text{Res } (G(z),\ 0) &= \left[\frac{d}{dz}\left(\frac{(z+4)}{2\left(z+\dfrac{1}{2}\right)(z-3)}\right)\right]_0 \\ &= \frac{1}{2}\left[\frac{d}{dz}\left(\frac{z+4}{z^2 - \dfrac{5}{2}z - \dfrac{3}{2}}\right)\right]_0 \end{aligned}$$

Task

Carry out the differentiation shown on the last line of the previous page, then substitute $z = 0$ and hence obtain the required residue.

Your solution

Answer

Differentiating by the quotient rule then substituting $z = 0$ gives

$$\text{Res}\ (G(z),\ 0) = \frac{17}{9}$$

Key Point 15

Residue at a Pole of Order k

If $G(z)$ has a k^{th} order pole at $z = z_0$

i.e. $G(z) = \dfrac{G_1(z)}{(z - z_0)^k} \qquad G_1(z_0) \neq 0$ and finite

$$\text{Res}\ (G(z),\ z_0) = \frac{1}{(k-1)!}\left[\frac{d^{k-1}}{dz^{k-1}}(G(z)\ (z - z_0)^k)\right]_{z_0} \qquad (21)$$

This formula reduces to (19) and (20) when $k = 1$ and 2 respectively.

Inverse z-transform formula

Recall that, by definition, the z-transform of a sequence $\{f_n\}$ is

$$F(z) = f_0 + f_1 z^{-1} + f_2 z^{-2} + \dots f_n z^{-n} + \dots$$

If we multiply both sides by z^{n-1} where n is a positive integer we obtain

$$F(z)z^{n-1} = f_0 z^{n-1} + f_1 z^{n-2} + f_2 z^{n-3} + \dots f_n z^{-1} + f_{n+1} z^{-2} + \dots$$

Using again a result from complex integration it can be shown from this expression that the general term f_n is given by

$$f_n = \text{sum of residues of } F(z)\ z^{n-1} \text{ at its poles} \qquad (22)$$

The poles of $F(z)z^{n-1}$ will be those of $F(z)$ with possibly additional poles at the origin.

To illustrate the residue method of inversion we shall re-do some of the earlier examples that were done using partial fractions.

Example:

$$Y(z) = \frac{z^2}{(z-a)(z-b)} \qquad a \neq b$$

so

$$Y(z)z^{n-1} = \frac{z^{n+1}}{(z-a)(z-b)} = G(z), \text{ say.}$$

$G(z)$ has first order poles at $z = a$, $z = b$ so using (19).

$$\begin{aligned} \text{Res } (G(z),\ a) &= \left[\frac{z^{n+1}}{z-b}\right]_a = \frac{a^{n+1}}{a-b} \\ \text{Res } (G(z),\ b) &= \left[\frac{z^{n+1}}{z-a}\right]_b = \frac{b^{n+1}}{b-a} = \frac{-b^{n+1}}{a-b} \end{aligned}$$

We need simply add these residues to obtain the required inverse z-transform

$$\therefore \qquad f_n = \frac{1}{(a-b)}(a^{n+1} - b^{n+1})$$

as before.

Task

Obtain, using (22), the inverse z-transform of

$$Y(z) = \frac{6z^2 - 9z}{(z-3)^2}$$

Firstly, obtain the pole(s) of $G(z) = Y(z)z^{n-1}$ and deduce the order:

Your solution

Answer

$$G(z) = Y(z)z^{n-1} = \frac{6z^{n+1} - 9z^n}{(z-3)^2}$$

whose only pole is one of second order at $z = 3$.

Now calculate the residue of $G(z)$ at $z = 3$ using (20) and hence write down the required inverse z-transform y_n:

Your solution

Answer

$$\begin{aligned}\text{Res } (G(z),\ 3) &= \left[\frac{d}{dz}(6z^{n+1} - 9z^n)\right]_3 \\ &= \left[6(n+1)z^n - 9nz^{n-1}\right]_3 \\ &= 6(n+1)3^n - 9n3^{n-1} \\ &= 6 \times 3^n + 3n3^n\end{aligned}$$

This is the same as was found by partial fractions, but there is considerably less labour by the residue method.

In the above examples all the poles of the various functions $G(z)$ were real. This is the easiest situation but the residue method will cope with complex poles.

Example

We showed earlier that

$$\frac{z^2}{z^2+1} \text{ and } \cos\left(\frac{n\pi}{2}\right)$$

formed a z-transform pair.

We will now obtain y_n if $Y(z) = \dfrac{z^2}{z^2+1}$ using residues.
Using residues with, from (22),

$$G(z) = \frac{z^{n+1}}{z^2+1} = \frac{z^{n+1}}{(z-\mathsf{i})(z+\mathsf{i})} \text{ where } \mathsf{i}^2 = -1.$$

we see that $G(z)$ has first order poles at the complex conjugate points $\pm$ i.
Using (19)

$$\text{Res } (G(z),\ i) = \left[\frac{z^{n+1}}{z+\mathsf{i}}\right]_{\mathsf{i}} = \frac{\mathsf{i}^{n+1}}{2\mathsf{i}} \qquad \text{Res } (G(z),\ -\mathsf{i}) = \frac{(-\mathsf{i})^{n+1}}{(-2\mathsf{i})}$$

(Note the complex conjugate residues at the complex conjugate poles.)
Hence $\mathbb{Z}^{-1}\{\dfrac{z^2}{z^2+1}\} = \dfrac{1}{2\mathsf{i}}\left(\mathsf{i}^{n+1} - (-\mathsf{i})^{n+1}\right)$

But $\mathsf{i} = e^{\mathsf{i}\pi/2}$ and $-\mathsf{i} = e^{-\mathsf{i}\pi/2}$, so the inverse z-transform is

$$\frac{1}{2\mathsf{i}}\left(e^{\mathsf{i}(n+1)\pi/2} - e^{-\mathsf{i}(n+1)\pi/2}\right) = \sin(n+1)\frac{\pi}{2} = \cos\left(\frac{n\pi}{2}\right) \qquad \text{as expected.}$$

Task

Show, using residues, that

$$\mathbb{Z}^{-1}\{\frac{z}{z^2+1}\} = \sin\left(\frac{n\pi}{2}\right)$$

Your solution

Answer

Using (22):

$$\begin{aligned}
G(z) &= z^{n-1}\frac{z}{z^2+1} = \frac{z^n}{z^2+1} = \frac{z^n}{(z+\mathrm{i})(z-\mathrm{i})} \\
\text{Res }(G(z),\ \mathrm{i}) &= \frac{\mathrm{i}^n}{2\mathrm{i}} \\
\text{Res }(G(z),\ -\mathrm{i}) &= \frac{(-\mathrm{i})^n}{-2\mathrm{i}} \\
\mathbb{Z}^{-1}\{\frac{z}{z^2+1}\} &= \frac{1}{2\mathrm{i}}(\mathrm{i}^n - (-\mathrm{i})^n) \\
&= \frac{1}{2\mathrm{i}}(e^{\mathrm{i}n\pi/2} - e^{-\mathrm{i}n\pi/2}) \\
&= \sin\left(\frac{n\pi}{2}\right)
\end{aligned}$$

4. An application of difference equations – currents in a ladder network

The application we will consider is that of finding the electric currents in each loop of the ladder resistance network shown, which consists of $(N+1)$ loops. The currents form a sequence $\{i_0, i_1, \ldots i_N\}$

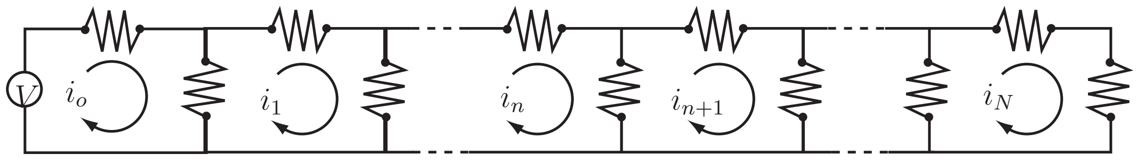


Figure 7

All the resistors have the same resistance R so loops 1 to N are identical. The zero'th loop contains an applied voltage V. In this zero'th loop, Kirchhoff's voltage law gives

$$V = Ri_0 + R(i_0 - i_1)$$

from which

$$i_1 = 2i_0 - \frac{V}{R} \qquad (23)$$

Similarly, applying the Kirchhoff law to the $(n+1)^{th}$ loop where there is no voltage source and 3 resistors

$$0 = Ri_{n+1} + R(i_{n+1} - i_{n+2}) + R(i_{n+1} - i_n)$$

from which

$$i_{n+2} - 3i_{n+1} + i_n = 0 \qquad n = 0, 1, 2, \ldots (N-2) \qquad (24)$$

(24) is the basic difference equation that has to be solved.

Task

Using the left shift theorems obtain the z-transform of equation (24). Denote by $I(z)$ the z-transform of $\{i_n\}$. Simplify the algebraic equation you obtain.

Your solution

Answer
We obtain

$$z^2I(z) - z^2i_0 - zi_1 - 3(zI(z) - zi_0) + I(z) = 0$$

Simplifying

$$(z^2 - 3z + 1)I(z) = z^2i_0 + zi_1 - 3zi_0 \qquad (25)$$

If we now eliminate i_1 using (23), the right-hand side of (25) becomes

$$z^2i_0 + z\left(2i_0 - \frac{V}{R}\right) - 3zi_0 = z^2i_0 - zi_0 - z\frac{V}{R} = i_0\left(z^2 - z - z\frac{V}{i_0R}\right)$$

Hence from (25)

$$I(z) = \frac{i_0\left(z^2 - \left(1 + \frac{V}{i_0R}\right)z\right)}{z^2 - 3z + 1} \qquad (26)$$

Our final task is to find the inverse z-transform of (26).

Task

Look at the table of z-transforms on page 35 (or at the back of the Workbook) and suggest what sequences are likely to arise by inverting $I(z)$ as given in (26).

Your solution

Answer
The most likely candidates are hyperbolic sequences because both $\{\cosh \alpha n\}$ and $\{\sinh \alpha n\}$ have z-transforms with denominator

$$z^2 - 2z\cosh\alpha + 1$$

which is of the same form as the denominator of (26), remembering that $\cosh\alpha \geq 1$. (Why are the trigonometric sequences $\{\cos \omega n\}$ and $\{\sin \omega n\}$ not plausible here?)

To proceed, we introduce a quantity α such that α is the positive solution of $2\cosh\alpha = 3$ from which (using $\cosh^2\alpha - \sinh^2\alpha \equiv 1$) we get

$$\sinh\alpha = \sqrt{\frac{9}{4} - 1} = \frac{\sqrt{5}}{2}$$

Hence (26) can be written

$$I(z) = i_0 \frac{\left(z^2 - \left(1 + \frac{V}{i_0 R}\right) z\right)}{z^2 - 2z\cosh\alpha + 1} \qquad (27)$$

To further progress, bearing in mind the z-transforms of $\{\cosh\alpha n\}$ and $\{\sinh\alpha n\}$, we must subtract and add $z\cosh\alpha$ to the numerator of (27), where $\cosh\alpha = \frac{3}{2}$.

$$\begin{aligned} I(z) &= i_0 \left[\frac{z^2 - z\cosh\alpha + \frac{3z}{2} - \left(1 + \frac{V}{i_0 R}\right) z}{z^2 - 2z\cosh\alpha + 1} \right] \\ &= i_0 \left[\frac{(z^2 - z\cosh\alpha)}{z^2 - 2z\cosh\alpha + 1} + \frac{\left(\frac{3}{2} - 1\right) z - \frac{Vz}{i_0 R}}{z^2 - 2z\cosh\alpha + 1} \right] \end{aligned}$$

The first term in the square bracket is the z-transform of $\{\cosh\alpha n\}$.

The second term is

$$\frac{\left(\frac{1}{2} - \frac{V}{i_0 R}\right) z}{z^2 - 2z\cosh\alpha + 1} = \frac{\left(\frac{1}{2} - \frac{V}{i_0 R}\right) \frac{2}{\sqrt{5}} z \frac{\sqrt{5}}{2}}{z^2 - 2z\cosh\alpha + 1}$$

which has inverse z-transform

$$\left(\frac{1}{2} - \frac{V}{i_0 R}\right) \frac{2}{\sqrt{5}} \sinh\alpha n$$

Hence we have for the loop currents

$$i_n = i_0 \cosh(\alpha n) + \left(\frac{i_0}{2} - \frac{V}{R}\right) \frac{2}{\sqrt{5}} \sinh(\alpha n) \qquad n = 0, 1, \ldots N \qquad (27)$$

where $\cosh\alpha = \dfrac{3}{2}$ determines the value of α.

Finally, by Kirchhoff's law applied to the rightmost loop

$$3i_N = i_{N-1}$$

from which, with (27), we could determine the value of i_0.

Exercises

1. Deduce the inverse z-transform of each of the following functions:

   (a) $\dfrac{2z^2 - 3z}{z^2 - 3z - 4}$

   (b) $\dfrac{2z^2 + z}{(z-1)^2}$

   (c) $\dfrac{2z^2 - z}{2z^2 - 2z + 2}$

   (d) $\dfrac{3z^2 + 5}{z^4}$

2. Use z-transforms to solve each of the following difference equations:

   (a) $y_{n+1} - 3y_n = 4^n \qquad y_0 = 0$

   (b) $y_n - 3y_{n-1} = 6 \qquad y_{-1} = 4$

   (c) $y_n - 2y_{n-1} = n \qquad y_{-1} = 0$

   (d) $y_{n+1} - 5y_n = 5^{n+1} \qquad y_0 = 0$

   (e) $y_{n+1} + 3y_n = 4\delta_{n-2} \qquad y_0 = 2$

   (f) $y_n - 7y_{n-1} + 10y_{n-2} = 0 \qquad y_{-1} = 16, \quad y_{-2} = 5$

   (g) $y_n - 6y_{n-1} + 9y_{n-2} = 0 \qquad y_{-1} = 1, \quad y_{-2} = 0$

Answers

1 (a) $(-1)^n + 4^n$ (b) $2 + 3n$ (c) $\cos(n\pi/3)$ (d) $3\delta_{n-2} + 5\delta_{n-4}$

2 (a) $y_n = 4^n - 3^n$ (b) $y_n = 21 \times 3^n - 3$ (c) $y_n = 2 \times 2^n - 2 - n$ (d) $y_n = n5^n$

(e) $y_n = 2 \times (-3)^n + 4 \times (-3)^{n-3}u_{n-2}$ (f) $y_n = 12 \times 2^n + 50 \times 5^n$ (g) $y_n = (6 + 3n)3^n$

Engineering Applications of z-Transforms

21.4

Introduction

In this Section we shall apply the basic theory of z-transforms to help us to obtain the response or output sequence for a discrete system. This will involve the concept of the transfer function and we shall also show how to obtain the transfer functions of series and feedback systems. We will also discuss an alternative technique for output calculations using convolution. Finally we shall discuss the initial and final value theorems of z-transforms which are important in digital control.

Prerequisites

Before starting this Section you should ...

- be familiar with basic z-transforms, particularly the shift properties

Learning Outcomes

On completion you should be able to ...

- obtain transfer functions for discrete systems including series and feedback combinations
- state the link between the convolution summation of two sequences and the product of their z-transforms

1. Applications of z-transforms

Transfer (or system) function

Consider a first order linear constant coefficient difference equation

$$y_n + a\ y_{n-1} = bx_n \qquad n = 0, 1, 2, \ldots \tag{1}$$

where $\{x_n\}$ is a given sequence.
Assume an initial condition y_{-1} is given.

Task

Take the z-transform of (1), insert the initial condition and obtain $Y(z)$ in terms of $X(z)$.

Your solution

Answer
Using the right shift theorem

$$Y(z) + a(z^{-1}Y(z) + y_{-1}) = b\ X(z)$$

where $X(z)$ is the z-transform of the given or **input** sequence $\{x_n\}$ and $Y(z)$ is the z-transform of the **response** or **output** sequence $\{y_n\}$.

Solving for $Y(z)$

$$Y(z)(1 + az^{-1}) = bX(z) - ay_{-1}$$

so

$$Y(z) = \frac{bX(z)}{1 + az^{-1}} - \frac{ay_{-1}}{1 + az^{-1}} \tag{2}$$

The form of (2) shows us clearly that $Y(z)$ is made up of two components, $Y_1(z)$ and $Y_2(z)$ say, where

(i) $Y_1(z) = \dfrac{bX(z)}{1 + az^{-1}}$ which depends on the input $X(z)$

(ii) $Y_2(z) = \dfrac{-ay_{-1}}{1 + az^{-1}}$ which depends on the initial condition y_{-1}.

Clearly, from (2), if $y_{-1} = 0$ (zero initial condition) then

$$Y(z) = Y_1(z)$$

and hence the term **zero-state response** is sometimes used for $Y_1(z)$.
Similarly if $\{x_n\}$ and hence $X(z) = 0$ (zero input)

$$Y(z) = Y_2(z)$$

and hence the term **zero-input** response can be used for $Y_2(z)$.

In engineering the difference equation (1) is regarded as modelling a **system** or more specifically a linear **discrete** time-invariant system. The terms linear and time-invariant arise because the difference equation (1) is linear and has constant coefficients i.e. the coefficients do not involve the index n. The term 'discrete' is used because sequences of numbers, not continuous quantities, are involved. As noted above, the given sequence $\{x_n\}$ is considered to be the input sequence and $\{y_n\}$, the solution to (1), is regarded as the output sequence.

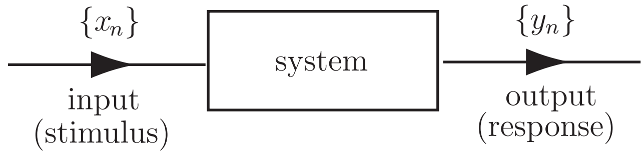


Figure 8

A more precise block diagram representation of a system can be easily drawn since only two operations are involved:

1. Multiplying the terms of a sequence by a constant.
2. Shifting to the right, or delaying, the terms of the sequence.

A system which consists of a single multiplier is denoted as shown by a triangular symbol:

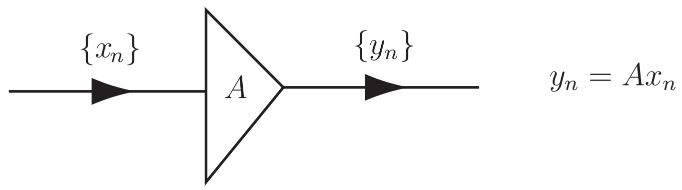


Figure 9

As we have seen earlier in this workbook a system which consists of only a single delay unit is represented symbolically as follows

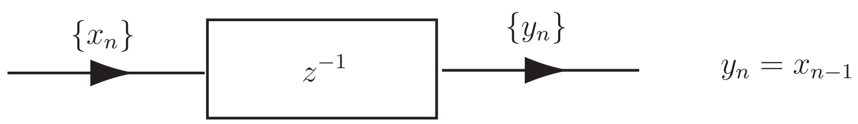


Figure 10

The system represented by the difference equation (1) consists of two multipliers and one delay unit. Because (1) can be written

$$y_n = bx_n - ay_{n-1}$$

a symbolic representation of (1) is as shown in Figure 11.

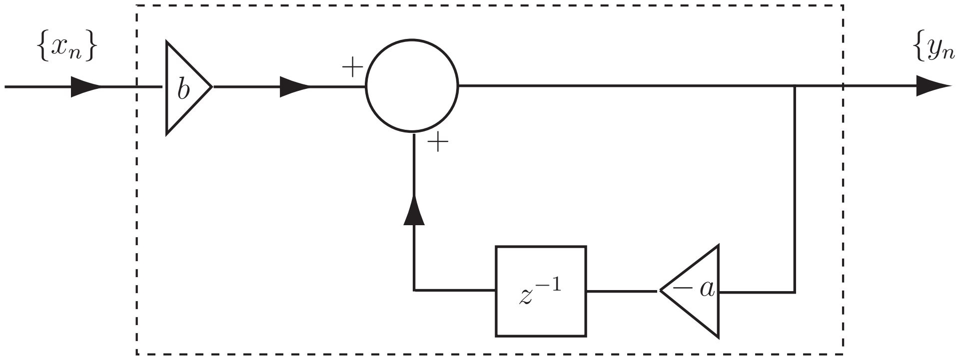


Figure 11

The circle symbol denotes an adder or summation unit whose output is the sum of the two (or more) sequences that are input to it.

We will now concentrate upon the zero state response of the system i.e. we will assume that the initial condition y_{-1} is zero.

Thus, using (2),

$$Y(z) = \frac{bX(z)}{1 + az^{-1}}$$

so

$$\frac{Y(z)}{X(z)} = \frac{b}{1 + az^{-1}} \qquad (3)$$

The quantity $\dfrac{Y(z)}{X(z)}$, the ratio of the output z-transform to the input z-transform, is called the **transfer function** of the discrete system. It is often denoted by $H(z)$.

Key Point 16

The transfer function $H(z)$ of a discrete system is defined by

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\text{z-transform of output sequence}}{\text{z-transform of input sequence}}$$

when the initial conditions are zero.

Task

(a) Write down the transfer function $H(z)$ of the system represented by (1)

(i) using negative powers of z
(ii) using positive powers of z.

(b) Write down the inverse z-transform of $H(z)$.

Your solution

Answer
(a) From (3)

(i) $H(z) = \dfrac{b}{1 + az^{-1}}$

(ii) $H(z) = \dfrac{bz}{z + a}$

(b) Referring to the Table of z-transforms at the end of the Workbook:

$\{h_n\} = b(-a)^n \qquad n = 0, 1, 2, \ldots$

We can represent any discrete system as follows

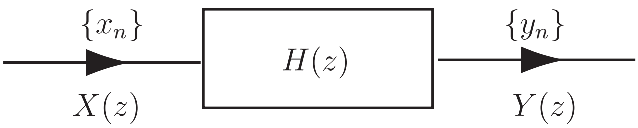


Figure 12

From the definition of the transfer function it follows that

$$Y(z) = X(z)H(z) \qquad \text{(at zero initial conditions).}$$

The corresponding relation between $\{y_n\}$, $\{x_n\}$ and the inverse z-transform $\{h_n\}$ of the transfer function will be discussed later; it is called a **convolution summation**.

The significance of $\{h_n\}$ is readily obtained.

Suppose $\qquad \{x_n\} = \begin{cases} 1 & n = 0 \\ 0 & n = 1, 2, 3, \ldots \end{cases}$

i.e. $\{x_n\}$ is the unit impulse sequence that is normally denoted by δ_n. Hence, in this case,

$$X(z) = Z\{\delta_n\} = 1 \quad \text{so} \quad Y(z) = H(z) \quad \text{and} \quad \{y_n\} = \{h_n\}$$

In words: $\{h_n\}$ is the response or output of a system where the input is the unit impulse sequence $\{\delta_n\}$. Hence $\{h_n\}$ is called the **unit impulse response** of the system.

Key Point 17

For a linear, time invariant discrete system, the unit impulse response and the system transfer function are a z-transform pair:

$$H(z) = \mathbb{Z}\{h_n\} \qquad \{h_n\} = \mathbb{Z}^{-1}\{H(z)\}$$

It follows from the previous Task that for the first order system (1)

$H(z) = \dfrac{b}{1 + az^{-1}} = \dfrac{bz}{z + a}$ is the transfer function and

$\{h_n\} = \{b(-a)^n\}$ is the unit impulse response sequence.

Task

Write down the transfer function of

(a) a single multiplier unit (b) a single delay unit.

Your solution

Answer

(a) $\{y_n\} = \{A\, x_n\}$ if the multiplying factor is A

$\therefore$ using the linearity property of z-transform

$Y(z) = AX(z)$

so $H(z) = \dfrac{Y(z)}{X(z)} = A$ is the required transfer function.

(b) $\{y_n\} = \{x_{n-1}\}$

so $Y(z) = z^{-1}X(z)$ (remembering that initial conditions are zero)

$\therefore$ $H(z) = z^{-1}$ is the transfer function of the single delay unit.

Task

Obtain the transfer function of the system.

$$y_n + a_1 y_{n-1} = b_0 x_n + b_1 x_{n-1} \qquad n = 0, 1, 2, \ldots$$

where $\{x_n\}$ is a known sequence with $x_n = 0$ for $n = -1, -2, \ldots$.
[Remember that the transfer function is only defined at zero initial condition i.e. assume $y_{-1} = 0$ also.]

Your solution

Answer
Taking z-transforms

$$\begin{aligned} Y(z) + a_1 z^{-1} Y(z) &= b_0 X(z) + b_1 z^{-1} X(z) \\ Y(z)(1 + a_1 z^{-1}) &= (b_0 + b_1 z^{-1}) X(z) \end{aligned}$$

so the transfer function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}} = \frac{b_0 z + b_1}{z + a_1}$$

Second order systems

Consider the system whose difference equation is

$$y_n + a_1 y_{n-1} + a_2 y_{n-2} = b x_n \qquad n = 0, 1, 2, \ldots \qquad (4)$$

where the input sequence $x_n = 0, \quad n = -1, -2, \ldots$

In exactly the same way as for first order systems it is easy to show that the system response has a z-transform with two components.

Task

Take the z-transform of (4), assuming given initial values $y_{-1},\ y_{-2}$. Show that $Y(z)$ has two components. Obtain the transfer function of the system (4).

Your solution

Answer

From (4)

$$Y(z) + a_1(z^{-1}Y(z) + y_{-1}) + a_2(z^{-2}Y(z) + z^{-1}y_{-1} + y_{-2}) = bX(z)$$
$$Y(z)(1 + a_1z^{-1} + a_2z^{-2}) + a_1y_{-1} + a_2z^{-1}y_{-1} + a_2y_{-2} = bX(z)$$

$$\therefore \qquad Y(z) = \frac{bX(z)}{1 + a_1z^{-1} + a_2z^{-2}} - \frac{(a_1y_{-1} + a_2z^{-1}y_{-1} + a_2y_{-2})}{1 + a_1z^{-1} + a_2z^{-2}} = Y_1(z) + Y_2(z) \quad \text{say.}$$

At zero initial conditions, $Y(z) = Y_1(z)$ so the transfer function is

$$H(z) = \frac{b}{1 + a_1z^{-1} + a_2z^{-2}} = \frac{bz^2}{z^2 + a_1z + a_2}.$$

Example

Obtain (i) the unit impulse response (ii) the unit step response of the system specified by the second order difference equation

$$y_n - \frac{3}{4}y_{n-1} + \frac{1}{8}y_{n-2} = x_n \qquad (5)$$

Note that both these responses refer to the case of zero initial conditions. Hence it is convenient to first obtain the transfer function $H(z)$ of the system and then use the relation $Y(z) = X(z)H(z)$ in each case.

We write down the transfer function of (5), using positive powers of z. Taking the z-transform of (5) at zero initial conditions we obtain

$$\begin{aligned} Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) &= X(z) \\ Y(z)\left(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right) &= X(z) \\ \therefore \qquad H(z) = \frac{Y(z)}{X(z)} &= \frac{z^2}{z^2 - \frac{3}{4}z + \frac{1}{8}} = \frac{z^2}{(z - \frac{1}{2})(z - \frac{1}{4})} \end{aligned}$$

We now complete the problem for inputs (i) $x_n = \delta_n$ (ii) $x_n = u_n$, the unit step sequence, using partial fractions.

$$H(z) = \frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)} = \frac{2z}{z-\frac{1}{2}} - \frac{z}{z-\frac{1}{4}}$$

(i) With $x_n = \delta_n$ so $X(z) = 1$ the response is, as we saw earlier,

$$Y(z) = H(z)$$

so $y_n = h_n$

where $h_n = \mathbb{Z}^{-1}H(z) = 2 \times \left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n \qquad n = 0, 1, 2, \ldots$

(ii) The z-transform of the unit step is $\dfrac{z}{z-1}$ so the unit step response has z-transform

$$\begin{aligned} Y(z) &= \frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)} \frac{z}{(z-1)} \\ &= -\frac{2z}{z-\frac{1}{2}} + \frac{\frac{1}{3}z}{z-\frac{1}{4}} + \frac{\frac{8}{3}z}{z-1} \end{aligned}$$

Hence, taking inverse z-transforms, the unit step response of the system is

$$y_n = (-2) \times \left(\frac{1}{2}\right)^n + \frac{1}{3} \times \left(\frac{1}{4}\right)^n + \frac{8}{3} \qquad n = 0, 1, 2, \ldots$$

Notice carefully the form of this unit step response - the first two terms decrease as n increases and are called **transients**. Thus

$$y_n \to \frac{8}{3} \quad \text{as} \quad n \to \infty$$

and the term $\dfrac{8}{3}$ is referred to as the **steady state** part of the unit step response.

Combinations of systems

The concept of transfer function enables us to readily analyse combinations of discrete systems.

Series combination

Suppose we have two systems S_1 and S_2 with transfer functions $H_1(z)$, $H_2(z)$ in series with each other. i.e. the output from S_1 is the input to S_2.

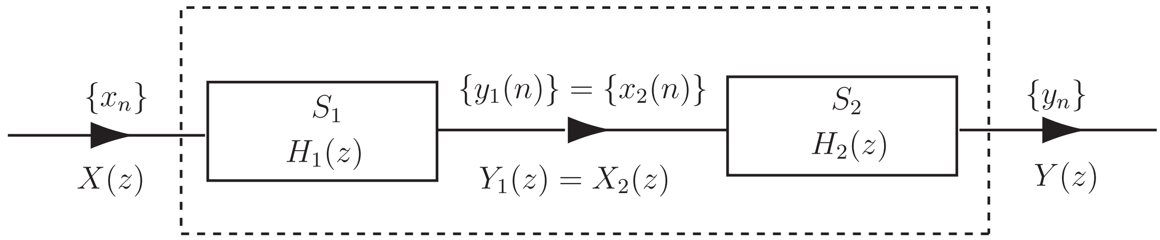


Figure 13

Clearly, at zero initial conditions,

$$\begin{aligned} Y_1(z) &= H_1(z)X(z) \\ Y(z) &= H_2(z)X_2(z) \\ &= H_2(z)Y_1(z) \\ \therefore \qquad Y(z) &= H_2(z)H_1(z)X(z) \end{aligned}$$

so the ratio of the final output transform to the input transform is

$$\frac{Y(z)}{X(z)} = H_2(z)\ H_1(z) \qquad (6)$$

i.e. the series system shown above is equivalent to a single system with transfer function $H_2(z)\ H_1(z)$

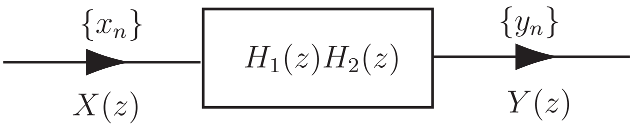


Figure 14

Task

Obtain (a) the transfer function (b) the governing difference equation of the system obtained by connecting two first order systems S_1 and S_2 in series. The governing equations are:

$$S_1: \quad y_n - ay_{n-1} = bx_n$$

$$S_2: \quad y_n - cy_{n-1} = dx_n$$

(a) Begin by finding the transfer function of S_1 and S_2 and then use (6):

Your solution

Answer

S_1: $\quad Y(z) - az^{-1}Y(z) = bX(z) \quad$ so $\quad H_1(z) = \dfrac{b}{1 - az^{-1}}$

S_2: $\quad H_2(z) = \dfrac{d}{1 - cz^{-1}}$

so the series arrangement has transfer function

$$\begin{aligned} H(z) &= \frac{bd}{(1 - az^{-1})(1 - cz^{-1})} \\ &= \frac{bd}{1 - (a+c)z^{-1} + acz^{-2}} \end{aligned}$$

If $X(z)$ and $Y(z)$ are the input and output transforms for the series arrangement, then

$$Y(z) = H(z)\ X(z) = \frac{bdX(z)}{1 - (a+c)z^{-1} + acz^{-2}}$$

(b) By transfering the denominator from the right-hand side to the left-hand side and taking inverse z-transforms obtain the required difference equation of the series arrangement:

Your solution

Answer

We have

$$Y(z)(1 - (a+c)z^{-1} + acz^{-2}) = bdX(z)$$

$$Y(z) - (a+c)z^{-1}Y(z) + acz^{-2}Y(z) = bdX(z)$$

from which, using the right shift theorem,

$$y_n - (a+c)y_{n-1} + acy_{n-2} = bd\ x_n.$$

which is the required difference equation.

You can see that the two first order systems in series have an equivalent second order system.

Feedback combination

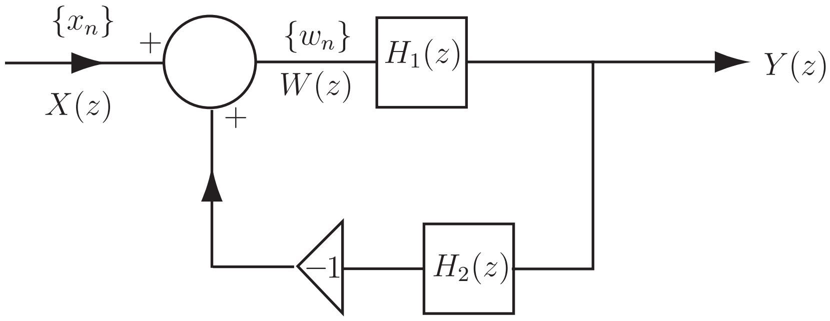


Figure 15

For the above **negative feedback** arrangement of two discrete systems with transfer functions $H_1(z),\ H_2(z)$ we have, at zero initial conditions,

$$Y(z) = W(z)H_1(z) \qquad \text{where} \qquad W(z) = X(z) - H_2(z)Y(z)$$

Task

Eliminate $W(z)$ and hence obtain the transfer function of the feedback system.

Your solution

Answer

$$\begin{aligned} Y(z) &= (X(z) - H_2(z)Y(z))H_1(z) \\ &= X(z)H_1(z) - H_2(z)H_1(z)Y(z) \end{aligned}$$

so

$$Y(z)(1 + H_2(z)H_1(z)) = X(z)H_1(z)$$

$$\therefore \qquad \frac{Y(z)}{X(z)} = \frac{H_1(z)}{1 + H_2(z)H_1(z)}$$

This is the required transfer function of the negative feedback system.

2. Convolution and z-transforms

Consider a discrete system with transfer function $H(z)$

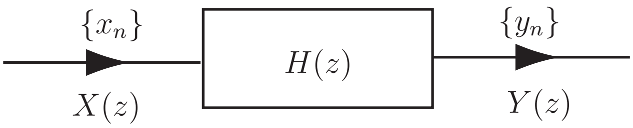


Figure 16

We know, from the definition of the transfer function that at zero initial conditions

$$Y(z) = X(z)H(z) \qquad (7)$$

We now investigate the corresponding relation between the input sequence $\{x_n\}$ and the output sequence $\{y_n\}$. We have seen earlier that the system itself can be characterised by its unit impulse response $\{h_n\}$ which is the inverse z-transform of $H(z)$.

We are thus seeking the inverse z-transform of the product $X(z)H(z)$. We emphasize immediately that this is **not** given by the product $\{x_n\}\{h_n\}$, a point we also made much earlier in the workbook. We go back to basic definitions of the z-transform:

$$Y(z) = y_0 + y_1z^{-1} + y_2z^{-2} + y_3z^{-3} + \dots$$

$$X(z) = x_0 + x_1z^{-1} + x_2z^{-2} + x_3z^{-3} + \dots$$

$$H(z) = h_0 + h_1z^{-1} + h_2z^{-2} + h_3z^{-3} + \dots$$

Hence, multiplying $X(z)$ by $H(z)$ we obtain, collecting the terms according to the powers of z^{-1}:

$$x_0h_0 + (x_0h_1 + x_1h_0)z^{-1} + (x_0h_2 + x_1h_1 + x_2h_0)z^{-2} + \dots$$

Task

Write out the terms in z^{-3} in the product $X(z)H(z)$ and, looking at the emerging pattern, deduce the coefficient of z^{-n}.

Your solution

Answer

$$(x_0h_3 + x_1h_2 + x_2h_1 + x_3h_0)z^{-3}$$

which suggests that the coefficient of z^{-n} is

$$x_0h_n + x_1h_{n-1} + x_2h_{n-2} + \ldots + x_{n-1}h_1 + x_nh_0$$

Hence, comparing corresponding terms in $Y(z)$ and $X(z)H(z)$

$$\left.\begin{array}{lll} z^0: & y_0 & = x_0h_0 \\ z^{-1}: & y_1 & = x_0h_1 + x_1h_0 \\ z^{-2}: & y_2 & = x_0h_2 + x_1h_1 + x_2h_0 \\ z^{-3}: & y_3 & = x_0h_3 + x_1h_2 + x_2h_1 + x_3h_0 \end{array}\right\} \qquad (8)$$

$$\vdots \qquad \vdots$$

$$z^{-n}: \quad y_n = x_0h_n + x_1h_{n-1} + x_2h_{n-2} + \ldots + x_{n-1}h_1 + x_nh_0 \qquad (9)$$

$$= \sum_{k=0}^{n} x_kh_{n-k} \qquad (10a)$$

$$= \sum_{k=0}^{n} h_kx_{n-k} \qquad (10b)$$

(Can you see why (10b) also follows from (9)?)

The sequence $\{y_n\}$ whose n th term is given by (9) and (10) is said to be the **convolution** (or more precisely the **convolution summation**) of the sequences $\{x_n\}$ and $\{h_n\}$,

The convolution of two sequences is usually denoted by an asterisk symbol ($*$).

We have shown therefore that

$$\mathbb{Z}^{-1}\{X(z)H(z)\} = \{x_n\} * \{h_n\} = \{h_n\} * \{x_n\}$$

where the general term of $\{x_n\} * \{h_n\}$ is in (10a) and that of $\{h_n\} * \{x_n\}$ is in (10b).

In words: the output sequence $\{y_n\}$ from a linear time invariant system is given by the convolution of the input sequence with the unit impulse response sequence of the system.

This result only holds if initial conditions are zero.

Key Point 18

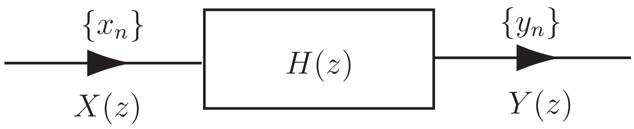


Figure 17

We have, at zero initial conditions

$Y(z) = X(z)H(z)$ (definition of transfer function)

$\{y_n\} = \{x_n\} * \{h_n\}$ (convolution summation)

where y_n is given in general by (9) and (10) with the first four terms written out explicitly in (8).

Although we have developed the convolution summation in the context of linear systems the proof given actually applies to any sequences i.e. for arbitrary causal sequences say $\{v_n\}$ $\{w_n\}$ with z-transforms $V(z)$ and $W(z)$ respectively:

$$\mathbb{Z}^{-1}\{V(z)W(z)\} = \{v_n\} * \{w_n\} \quad \text{or, equivalently,} \quad \mathbb{Z}(\{v_n\} * \{w_n\}) = V(z)W(z).$$

Indeed it is simple to prove this second result from the definition of the z-transform for any causal sequences $\{v_n\} = \{v_0, v_1, v_2, \ldots\}$ and $\{w_n\} = \{w_0, w_1, w_2, \ldots\}$

Thus since the general term of $\{v_n\} * \{w_n\}$ is $\displaystyle\sum_{k=0}^{n} v_k w_{n-k}$

we have

$$\mathbb{Z}(\{v_n\} * \{w_n\}) = \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n} v_k w_{n-k}\right\} z^{-n}$$

or, since $w_{n-k} = 0$ if $k > n$,

$$\mathbb{Z}(\{v_n\} * \{w_n\}) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} v_k w_{n-k} z^{-n}$$

Putting $m = n - k$ or $n = m + k$ we obtain

$$\mathbb{Z}(\{v_n\} * \{w_n\}) = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} v_k w_m z^{-(m+k)} \qquad \text{(Why is the lower limit } m = 0 \text{ correct?)}$$

Finally,

$$\mathbb{Z}(\{v_n\} * \{w_n\}) = \sum_{m=0}^{\infty} w_m z^{-m} \sum_{k=0}^{\infty} v_k z^{-k} = W(z)V(z)$$

which completes the proof.

Example 2

Calculate the convolution $\{y_n\}$ of the sequences

$$\{v_n\} = \{a^n\} \qquad \{w_n\} = \{b^n\} \qquad a \neq b$$

(i) directly (ii) using z-transforms.

Solution

(i) We have from (10)

$$\begin{aligned} y_n &= \sum_{k=0}^{n} v_k w_{n-k} = \sum_{k=0}^{n} a^k b^{n-k} \\ &= b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k \\ &= b^n \left(1 + \left(\frac{a}{b}\right) + \left(\frac{a}{b}\right)^2 + \dots \left(\frac{a}{b}\right)^n\right) \end{aligned}$$

The bracketed sum involves $n+1$ terms of a geometric series of common ratio $\dfrac{a}{b}$.

$$\begin{aligned} \therefore \qquad y_n &= b^n \frac{\left(1 - \left(\frac{a}{b}\right)^{n+1}\right)}{1 - \frac{a}{b}} \\ &= \frac{(b^{n+1} - a^{n+1})}{(b-a)} \end{aligned}$$

(ii) The z-transforms are

$$V(z) = \frac{z}{z-a}$$

$$W(z) = \frac{z}{z-b}$$

so

$$\begin{aligned} \therefore \qquad y_n &= \mathbb{Z}^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\} \\ &= \frac{b^{n+1} - a^{n+1}}{(b-a)} \qquad \text{using partial fractions or residues} \end{aligned}$$

Task

Obtain by two methods the convolution of the causal sequence

$\{2^n\} = \{1, 2, 2^2, 2^3, \ldots\}$

with itself.

Your solution

Answer

(a) By direct use of (10) if $\{y_n\} = \{2^n\} * \{2^n\}$

$$y_n = \sum_{k=0}^{n} 2^k 2^{n-k} = 2^n \sum_{k=0}^{n} 1 = (n+1)2^n$$

(b) Using z-transforms:

$$\mathbb{Z}\{2^n\} = \frac{z}{z-2}$$

so $\qquad \{y_n\} = \mathbb{Z}^{-1}\{\dfrac{z^2}{(z-2)^2}\}$

We will find this using the residue method. $Y(z)z^{n-1}$ has a second order pole at $z = 2$.

$$\begin{aligned} \therefore \qquad y_n &= \text{Res}\left(\frac{z^{n+1}}{(z-2)^2},\ 2\right) \\ &= \left[\frac{d}{dz}z^{n+1}\right]_2 = (n+1)2^n \end{aligned}$$

3. Initial and final value theorems of z-transforms

These results are important in, for example, Digital Control Theory where we are sometimes particularly interested in the initial and ultimate behaviour of systems.

Initial value theorem

.

If f_n is a sequence with z-transform $F(z)$ then the 'initial value' f_0 is given by

$$f_0 = \lim_{z\to\infty} F(z) \qquad \text{(provided, of course, that this limit exists).}$$

This result follows, at least informally, from the definition of the z-transform:

$$F(z) = f_0 + f_1 z^{-1} + f_2 z^{-2} + \dots$$

from which, taking limits as $z \to \infty$ the required result is obtained.

Task

Obtain the z-transform of

$$f(n) = 1 - a^n, \qquad 0 < a < 1$$

Verify the initial value theorem for the z-transform pair you obtain.

Your solution

Answer

Using standard z-transforms we obtain

$$\begin{aligned} Z\{f_n\} = F(z) &= \frac{z}{z-1} - \frac{z}{z-a} \\ &= \frac{1}{1-z^{-1}} - \frac{1}{1-az^{-1}} \end{aligned}$$

hence, as $z \to \infty$: $F(z) \to 1 - 1 = 0$

Similarly, as $n \to 0$

$$f_n \to 1 - 1 = 0$$

so the initial value theorem is verified for this case.

Final value theorem

Suppose again that $\{f_n\}$ is a sequence with z-transform $F(z)$. We further assume that all the poles of $F(z)$ lie inside the unit circle in the $z-$plane (i.e. have magnitude less than 1) apart possibly from a first order pole at $z = 1$.

The 'final value' of f_n i.e. $\lim_{n\to\infty} f_n$ is then given by $\quad \lim_{n\to\infty} f_n = \lim_{z\to 1}(1 - z^{-1})F(z)$

Proof: Recalling the left shift property

$$Z\{f_{n+1}\} = zF(z) - zf_0$$

we have

$$Z\{f_{n+1} - f_n\} = \lim_{k\to\infty} \sum_{n=0}^{k} (f_{n+1} - f_n)z^{-n} = zF(z) - zf_0 - F(z)$$

or, alternatively, dividing through by z on both sides:

$$(1 - z^{-1})F(z) - f_0 = \lim_{k\to\infty} \sum_{n=0}^{k} (f_{n+1} - f_n)z^{-(n+1)}$$

Hence $\quad (1 - z^{-1})F(z) = f_0 + (f_1 - f_0)z^{-1} + (f_2 - f_1)z^{-2} + \dots$

or as $z \to 1$

$$\begin{aligned} \lim_{z\to 1}(1 - z^{-1})F(z) &= f_0 + (f_1 - f_0) + (f_2 - f_1) + \dots \\ &= \lim_{k\to\infty} f_k \end{aligned}$$

Example

Again consider the sequence $\quad f_n = 1 - a^n \qquad 0 < a < 1 \quad$ and its z-transform

$$F(z) = \frac{z}{z-1} - \frac{z}{z-a} = \frac{1}{1-z^{-1}} - \frac{1}{1-az^{-1}}$$

Clearly as $n \to \infty$ then $f_n \to 1$.
Considering the right-hand side

$$(1 - z^{-1})F(z) = 1 - \frac{(1 - z^{-1})}{1 - az^{-1}} \to 1 - 0 = 1 \quad \text{as } z \to 1.$$

Note carefully that

$$F(z) = \frac{z}{z-1} - \frac{z}{z-a}$$

has a pole at a $(0 < a < 1)$ and a simple pole at $z = 1$.

The final value theorem does not hold for z-transform poles outside the unit circle
e.g. $\qquad f_n = 2^n \qquad F(z) = \dfrac{z}{z-2}$
Clearly $f_n \to \infty$ as $n \to \infty$
whereas

$$(1 - z^{-1})F(z) = \left(\frac{z-1}{z}\right)\frac{z}{(z-2)} \qquad \to 0 \text{ as } z \to 1$$

Exercises

1. A low pass digital filter is characterised by

$$y_n = 0.1x_n + 0.9y_{n-1}$$

Two such filters are connected in series. Deduce the transfer function and governing difference equation for the overall system. Obtain the response of the series system to (i) a unit step and (ii) a unit alternating input. Discuss your results.

2. The two systems

$$y_n = x_n - 0.7x_{n-1} + 0.4y_{n-1}$$

$$y_n = 0.9x_{n-1} - 0.7y_{n-1}$$

are connected in series. Find the difference equation governing the overall system.

3. A system S_1 is governed by the difference equation

$$y_n = 6x_{n-1} + 5y_{n-1}$$

It is desired to stabilise S_1 by using a feedback configuration. The system S_2 in the feedback loop is characterised by

$$y_n = \alpha x_{n-1} + \beta y_{n-1}$$

Show that the feedback system S_3 has an overall transfer function

$$H_3(z) = \frac{H_1(z)}{1 + H_1(z)H_2(z)}$$

and determine values for the parameters α and β if $H_3(z)$ is to have a second order pole at $z = 0.5$. Show briefly why the feedback systems S_3 *stabilizes* the original system.

4. Use z-transforms to find the sum of squares of all integers from 1 to n:

$$y_n = \sum_{k=1}^{n} k^2$$

[Hint: $y_n - y_{n-1} = n^2$]

5. Evaluate each of the following convolution summations (i) directly (ii) using z-transforms:

(a) $a^n * b^n \quad a \neq b$ (b) $a^n * a^n$ (c) $\delta_{n-3} * \delta_{n-5}$

(d) $x_n * x_n$ where $x_n = \begin{cases} 1 & n = 0, 1, 2, 3 \\ 0 & n = 4, 5, 6, 7 \ldots \end{cases}$

Answers

1. Step response: $y_n = 1 - (0.99)(0.9)^n - 0.09n(0.9)^n$

   Alternating response: $y_n = \dfrac{1}{361}(-1)^n + \dfrac{2.61}{361}(0.9)^n + \dfrac{1.71}{361}n(0.9)^n$

2. $y_n + 0.3y_{n-1} - 0.28y_{n-2} = 0.9x_{n-1} - 0.63x_{n-2}$

3. $\alpha = 3.375 \qquad \beta = -4$

4. $\displaystyle\sum_{k=1}^{n} k^2 = \frac{(2n+1)(n+1)n}{6}$

5. (a) $\dfrac{1}{(a-b)}(a^{n+1} - b^{n+1})$ (b) $(n+1)a^n$ (c) δ_{n-8} (d) $\{1, 2, 3, 4, 3, 2, 1\}$

Sampled Functions 21.5

Introduction

A sequence can be obtained by **sampling** a continuous function or signal and in this Section we show first of all how to extend our knowledge of z-transforms so as to be able to deal with sampled signals. We then show how the z-transform of a sampled signal is related to the Laplace transform of the unsampled version of the signal.

Prerequisites

Before starting this Section you should . . .

- possess an outline knowledge of Laplace transforms and of z-transforms

Learning Outcomes

On completion you should be able to . . .

- take the z-transform of a sequence obtained by sampling
- state the relation between the z-transform of a sequence obtained by sampling and the Laplace transform of the underlying continuous signal

1. Sampling theory

If a **continuous-time** signal $f(t)$ is sampled at terms $t = 0, T, 2T, \ldots nT, \ldots$ then a sequence of values

$$\{f(0),\ f(T),\ f(2T), \ldots f(nT), \ldots\}$$

is obtained. The quantity T is called the **sample interval** or **sample period**.

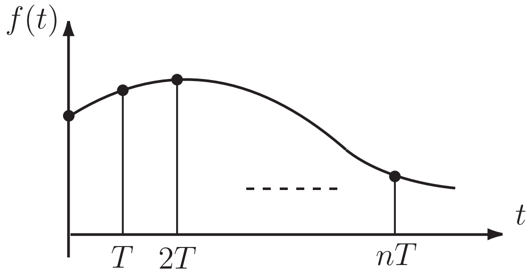


Figure 18

In the previous Sections of this Workbook we have used the simpler notation $\{f_n\}$ to denote a sequence. If the sequence has actually arisen by sampling then f_n is just a convenient notation for the sample value $f(nT)$.

Most of our previous results for z-transforms of sequences hold with only minor changes for sampled signals.

So consider a continuous signal $f(t)$; its z-transform is the z-transform of the sequence of sample values i.e.

$$\mathbb{Z}\{f(t)\} = \mathbb{Z}\{f(nT)\} = \sum_{n=0}^{\infty} f(nT)z^{-n}$$

We shall briefly obtain z-transforms of common sampled signals utilizing results obtained earlier. You may assume that all signals are sampled at $0, T, 2T, \ldots nT, \ldots$

Unit step function

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Since the sampled values here are a sequence of 1's,

$$\begin{aligned} \mathbb{Z}\{u(t)\} = \mathbb{Z}\{u_n\} &= \frac{1}{1 - z^{-1}} \\ &= \frac{z}{z-1} \qquad |z| > 1 \end{aligned}$$

where $\{u_n\} = \{\underset{\uparrow}{1},\ 1,\ 1,\ \ldots\}$ is the unit step sequence.

Ramp function

$$r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

The sample values here are

$$\{r(nT)\} = \{0,\ T,\ 2T,\ \ldots\}$$

The ramp sequence $\{r_n\} = \{0,\ 1,\ 2,\ \ldots\}$ has z-transform $\dfrac{z}{(z-1)^2}$.

Hence $\mathbb{Z}\{r(nT)\} = \dfrac{Tz}{(z-1)^2}$ since $\{r(nT)\} = T\{r_n\}$.

Task

Obtain the z-transform of the exponential signal

$$f(t) = \begin{cases} e^{-\alpha t} & t \geq 0 \\ 0 & t < 0. \end{cases}$$

[Hint: use the z-transform of the geometric sequence $\{a^n\}$.]

Your solution

Answer

The sample values of the exponential are

$$\{1,\ e^{-\alpha T},\ e^{-\alpha 2T}, \ldots, e^{-\alpha nT}, \ldots\}$$

i.e. $f(nT) = e^{-\alpha nT} = (e^{-\alpha T})^n$.

But $\mathbb{Z}\{a^n\} = \dfrac{z}{z-a}$

$$\therefore \qquad \mathbb{Z}\{(e^{-\alpha T})^n\} = \frac{z}{z - e^{-\alpha T}} = \frac{1}{1 - e^{-\alpha T} z^{-1}}$$

Sampled sinusoids

Earlier in this Workbook we obtained the z-transform of the sequence $\{\cos \omega n\}$ i.e.

$$\mathbb{Z}\{\cos \omega n\} = \frac{z^2 - z\cos\omega}{z^2 - 2z\cos\omega + 1}$$

Hence, since sampling the continuous sinusoid

$$f(t) = \cos \omega t$$

yields the sequence $\{\cos n\omega T\}$ we have, simply replacing ω by ωT in the z-transform:

$$\begin{aligned} \mathbb{Z}\{\cos \omega t\} &= \mathbb{Z}\{\cos n\omega T\} \\ &= \frac{z^2 - z\cos\omega T}{z^2 - 2z\cos\omega T + 1} \end{aligned}$$

Obtain the z-transform of the sampled version of the sine wave $f(t) = \sin \omega t$.

Your solution

Answer

$$\mathbb{Z}\{\sin \omega n\} = \frac{z\sin\omega}{z^2 - 2z\cos\omega + 1}$$

$$\begin{aligned} \therefore \qquad \mathbb{Z}\{\sin \omega t\} &= \mathbb{Z}\{\sin n\omega T\} \\ &= \frac{z\sin\omega T}{z^2 - 2z\cos\omega T + 1} \end{aligned}$$

Shift theorems

These are similar to those discussed earlier in this Workbook but for sampled signals the shifts are by integer multiples of the sample period T. For example a simple right shift, or delay, of a sampled signal by one sample period is shown in the following figure:

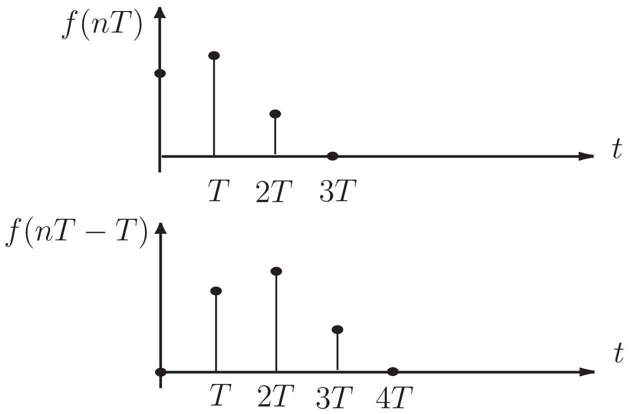


Figure 19

The right shift properties of z-transforms can be written down immediately. (Look back at the shift properties in Section 21.2 subsection 5, if necessary:)

If $y(t)$ has z-transform $Y(z)$ which, as we have seen, really means that its sample values $\{y(nT)\}$ give $Y(z)$, then for $y(t)$ shifted to the right by one sample interval the z-transform becomes

$$\mathbb{Z}\{y(t-T)\} = y(-T) + z^{-1}Y(z)$$

The proof is very similar to that used for sequences earlier which gave the result:

$$\mathbb{Z}\{y_{n-1}\} = y_{-1} + z^{-1}Y(z)$$

Task

Using the result

$$\mathbb{Z}\{y_{n-2}\} = y_{-2} + y_{-1}z^{-1} + z^{-2}Y(z)$$

write down the result for $\mathbb{Z}\{y(t-2T)\}$

Your solution

Answer

$$\mathbb{Z}\{y(t-2T)\} = y(-2T) + y(-T)z^{-1} + z^{-2}Y(z)$$

These results can of course be generalised to obtain $\mathbb{Z}\{y(t-mT)\}$ where m is any positive integer. In particular, for causal or one-sided signals $y(t)$ (i.e. signals which are zero for $t<0$):

$$\mathbb{Z}\{y(t-mT)\} = z^{-m}Y(z)$$

Note carefully here that the power of z is still z^{-m} **not** z^{-mT}.

Examples:
For the unit step function we saw that:

$$\mathbb{Z}\{u(t)\} = \frac{z}{z-1} = \frac{1}{1-z^{-1}}$$

Hence from the shift properties above we have immediately, since $u(t)$ is certainly causal,

$$\begin{aligned}
\mathbb{Z}\{u(t-T)\} &= \frac{zz^{-1}}{z-1} = \frac{z^{-1}}{1-z^{-1}} \\
\mathbb{Z}\{u(t-3T)\} &= \frac{zz^{-3}}{z-1} = \frac{z^{-3}}{1-z^{-1}}
\end{aligned}$$

and so on.

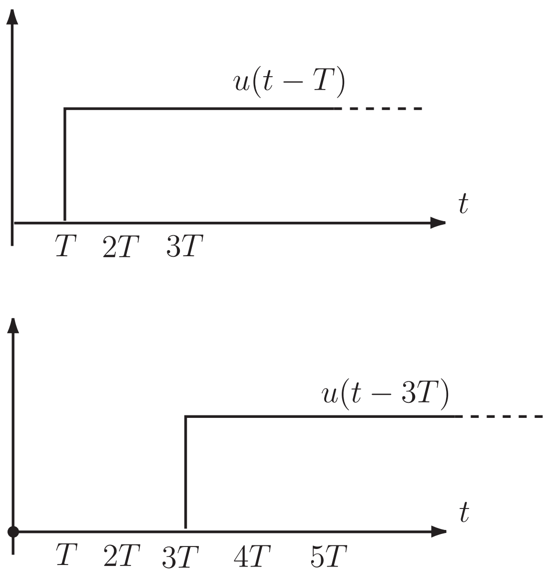


Figure 20

2. z-transforms and Laplace transforms

In this Workbook we have developed the theory and some applications of the z-transform from first principles. We mentioned much earlier that the z-transform plays essentially the same role for discrete systems that the Laplace transform does for continuous systems. We now explore the precise link between these two transforms. A brief knowledge of Laplace transform will be assumed.

At first sight it is not obvious that there is a connection. The z-transform is a **summation** defined, for a sampled signal $f_n \equiv f(nT)$, as

$$F(z) = \sum_{n=0}^{\infty} f(nT)z^{-n}$$

while the Laplace transform written symbolically as $\mathbb{L}\{f(t)\}$ is an **integral**, defined for a continuous time function $f(t),\ t \geq 0$ as

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt.$$

Thus, for example, if

$$f(t) \;=\; e^{-\alpha t} \qquad \text{(continuous time exponential)}$$

$$\mathbb{L}\{f(t)\} = F(s) \;=\; \frac{1}{s+\alpha}$$

which has a (simple) pole at $s = -\alpha = s_1$ say.
As we have seen, sampling $f(t)$ gives the sequence $\{f(nT)\} = \{e^{-\alpha nT}\}$ with z-transform

$$F(z) = \frac{1}{1 - e^{-\alpha T}z^{-1}} = \frac{z}{z - e^{-\alpha T}}.$$

The z-transform has a pole when $z = z_1$ where

$$z_1 = e^{-\alpha T} = e^{s_1 T}$$

[Note the abuse of notations in writing both $F(s)$ and $F(z)$ here since in fact these are **different** functions.]

Task

The continuous time function $f(t) = te^{-\alpha t}$ has Laplace transform $F(s) = \dfrac{1}{(s+\alpha)^2}$

Firstly write down the pole of this function and its order:

Your solution

Answer

$F(s) = \dfrac{1}{(s+\alpha)^2}$ has its pole at $s = s_1 = -\alpha$. The pole is second order.

Now obtain the z-transform $F(z)$ of the sampled version of $f(t)$, locate the pole(s) of $F(z)$ and state the order:

Your solution

Answer

Consider $f(nT) = nTe^{-\alpha nT} = (nT)(e^{-\alpha T})^n$

The ramp sequence $\{nT\}$ has z-transform $\dfrac{Tz}{(z-1)^2}$

$\therefore \quad f(nT)$ has z-transform

$$F(z) = \frac{Tze^{\alpha T}}{(ze^{\alpha T}-1)^2} = \frac{Tze^{-\alpha T}}{(z-e^{-\alpha T})^2} \qquad \text{(see Key Point 8)}$$

This has a (second order) pole when $z = z_1 = e^{-\alpha T} = e^{s_1 T}$.

We have seen in both the above examples a close link between the pole s_1 of the Laplace transform of $f(t)$ and the pole z_1 of the z-transform of the sampled version of $f(t)$ i.e.

$$z_1 = e^{s_1 T} \qquad (1)$$

where T is the sample interval.

Multiple poles lead to similar results i.e. if $F(s)$ has poles $s_1, s_2, \ldots$ then $F(z)$ has poles $z_1, z_2, \ldots$ where $z_i = e^{s_i T}$.

The relation (1) between the poles is, in fact, an example of a more general relation between the values of s and z as we shall now investigate.

Key Point 19

The **unit impulse function** $\delta(t)$ can be defined informally as follows:

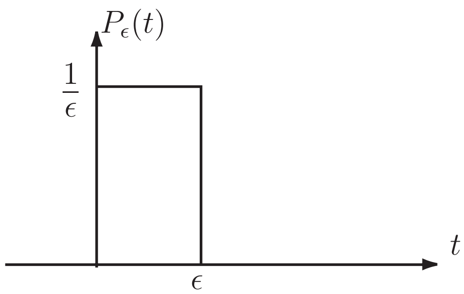


Figure 21

The rectangular pulse $P_\epsilon(t)$ of width ε and height $\dfrac{1}{\varepsilon}$ shown in Figure 21 encloses unit area and has Laplace transform

$$P_\varepsilon(s) = \int_0^\varepsilon \frac{1}{\varepsilon}e^{-st} = \frac{1}{\varepsilon s}(1-e^{-\varepsilon s}) \qquad (2)$$

As ε becomes smaller $P_\varepsilon(t)$ becomes taller and narrower but still encloses unit area. The unit impulse function $\delta(t)$ (sometimes called the Dirac delta function) can be defined as

$$\delta(t) = \lim_{\varepsilon \to 0} P_\varepsilon(t)$$

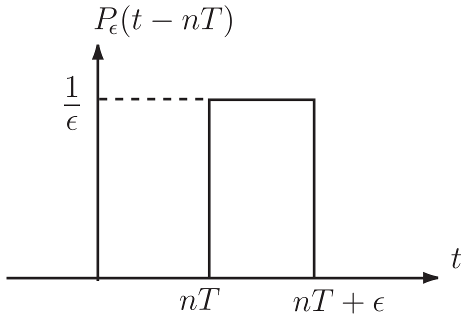
The Laplace transform, say $\Delta(s)$, of $\delta(t)$ can be obtained correspondingly by letting $\epsilon \to 0$ in (2), i.e.

$$\begin{aligned}
\Delta(s) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon s}(1 - e^{-\varepsilon s}) \\
&= \lim_{\varepsilon \to 0} \frac{1 - (1 - \varepsilon s + \frac{(\varepsilon s)^2}{2!} - \ldots)}{\varepsilon s} \qquad \text{(Using the Maclaurin seies expansion of } e^{-\varepsilon s}\text{)} \\
&= \lim_{\varepsilon \to 0} \frac{\varepsilon s - \frac{(\varepsilon s)^2}{2!} + \frac{(\varepsilon s)^3}{3!} + \ldots}{\varepsilon s} \\
&= 1
\end{aligned}$$

i.e. $\quad \mathbb{L}\delta(t) = 1 \qquad\qquad (3)$

Task

A shifted unit impulse $\delta(t - nT)$ is defined as $\lim_{\varepsilon \to 0} P_\varepsilon(t - nT)$ as illustrated below.



Obtain the Laplace transform of this rectangular pulse and, by letting $\varepsilon \to 0$, obtain the Laplace transform of $\delta(t - nT)$.

Your solution

Answer

$$\begin{aligned}
\mathbb{L}\{P_\varepsilon(t-nT)\} = \int_{nT}^{nT+\varepsilon} \frac{1}{\varepsilon} e^{-st} dt \quad &= \quad \frac{1}{\varepsilon s} \Big[-e^{-st} \Big]_{nT}^{nT+\varepsilon} \\
&= \quad \frac{1}{\varepsilon s} \left(e^{-snT} - e^{-s(nT+\varepsilon)} \right) \\
&= \quad \frac{1}{\varepsilon s} e^{-snT}(1-e^{-s\varepsilon}) \to e^{-snT} \quad \text{as } \varepsilon \to 0
\end{aligned}$$

Hence $\mathbb{L}\{\delta(t-nT)\} = e^{-snT}$ (4)

which reduces to the result (3)

$\mathbb{L}\{\delta(t)\} = 1$ when $n = 0$

These results (3) and (4) can be compared with the results

$$\mathbb{Z}\{\delta_n\} = 1$$

$$\mathbb{Z}\{\delta_{n-m}\} = z^{-m}$$

for discrete impulses of height 1.

Now consider a continuous function $f(t)$. Suppose, as usual, that this function is sampled at $t = nT$ for $n = 0, 1, 2, \ldots$

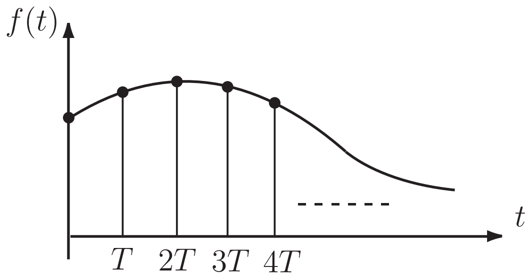


Figure 22

This sampled equivalent of $f(t)$, say $f_*(t)$ can be defined as a sequence of equidistant impulses, the 'strength' of each impulse being the sample value $f(nT)$i.e.

$$f_*(t) = \sum_{n=0}^{\infty} f(nT)\delta(t-nT)$$

This function is a continuous-time signal i.e. is defined for all t. Using (4) it has a Laplace transform

$$F_*(s) = \sum_{n=0}^{\infty} f(nT)e^{-snT} \qquad (5)$$

If, in this sum (5) we replace e^{sT} by z we obtain the z-transform of the sequence $\{f(nT)\}$ of samples:

$$\sum_{n=0}^{\infty} f(nT)z^{-n}$$

Key Point 20

The Laplace transform

$$F(s) = \sum_{n=0}^{\infty} f(nT)e^{-snT}$$

of a sampled function is equivalent to the z-transform $F(z)$ of the sequence $\{f(nT)\}$ of sample values with $z = e^{sT}$.

Table 2: z-transforms of some sampled signals

This table can be compared with the table of the z-transforms of sequences on the following page.

$f(t)$ $t \geq 0$	$f(nT)$ $n = 0, 1, 2, \ldots$	$F(z)$	Radius of convergence R
1	1	$\dfrac{z}{z-1}$	1
t	nT	$\dfrac{z}{(z-1)^2}$	1
t^2	$(nT)^2$	$\dfrac{T^2 z(z+1)}{(z-1)^3}$	1
$e^{-\alpha t}$	$e^{-\alpha nT}$	$\dfrac{z}{z - e^{-\alpha T}}$	$\|e^{-\alpha T}\|$
$\sin \omega t$	$\sin n\omega T$	$\dfrac{z \sin \omega T}{z^2 - 2z\cos\omega T + 1}$	1
$\cos \omega t$	$\cos n\omega T$	$\dfrac{z(z - \cos\omega T)}{z^2 - 2z\cos\omega T + 1}$	1
$te^{-\alpha t}$	$nTe^{-\alpha nT}$	$\dfrac{Tze^{-\alpha T}}{(z - e^{-\alpha T})^2}$	$\|e^{-\alpha T}\|$
$e^{-\alpha t} \sin \omega t$	$e^{-\alpha nT} \sin \omega nT$	$\dfrac{e^{-\alpha T} z^{-1} \sin\omega T}{1 - 2e^{-\alpha T} z^{-1} \cos\omega T + e^{-2\alpha T} z^{-2}}$	$\|e^{-\alpha T}\|$
$e^{-\alpha T} \cos \omega t$	$e^{-\alpha nT} \cos \omega nT$	$\dfrac{1 - e^{-\alpha T} z^{-1} \cos\omega T}{1 - 2e^{-\alpha T} z^{-1} \cos\omega T + e^{-2\alpha T} z^{-2}}$	$\|e^{-\alpha T}\|$

Note: R is such that the closed forms of $F(z)$ (those listed in the above table) are valid for $|z| > R$.

Table of z-transforms

f_n	$F(z)$	Name
δ_n	1	unit impulse
δ_{n-m}	z^{-m}	
u_n	$\dfrac{z}{z-1}$	unit step sequence
a^n	$\dfrac{z}{z-a}$	geometric sequence
$e^{\alpha n}$	$\dfrac{z}{z-e^{\alpha}}$	
$\sinh \alpha n$	$\dfrac{z \sinh \alpha}{z^2 - 2z\cosh\alpha + 1}$	
$\cosh \alpha n$	$\dfrac{z^2 - z \cosh \alpha}{z^2 - 2z\cosh\alpha + 1}$	
$\sin \omega n$	$\dfrac{z \sin \omega}{z^2 - 2z\cos\omega + 1}$	
$\cos \omega n$	$\dfrac{z^2 - z \cos \omega}{z^2 - 2z\cos\omega + 1}$	
$e^{-\alpha n} \sin \omega n$	$\dfrac{z e^{-\alpha} \sin \omega}{z^2 - 2ze^{-\alpha}\cos\omega + e^{-2\alpha}}$	
$e^{-\alpha n} \cos \omega n$	$\dfrac{z^2 - z e^{-\alpha} \cos \omega}{z^2 - 2ze^{-\alpha}\cos\omega + e^{-2\alpha}}$	
n	$\dfrac{z}{(z-1)^2}$	ramp sequence
n^2	$\dfrac{z(z+1)}{(z-1)^3}$	
n^3	$\dfrac{z(z^2+4z+1)}{(z-1)^4}$	
$a^n f_n$	$F\left(\dfrac{z}{a}\right)$	
$n\ f_n$	$-z\dfrac{dF}{dz}$	